MST 204.18

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods.

mathematical models and methods

Unit 18

Polynomial approximations

The Open University

Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

Unit 18
Polynomial approximations

Prepared for the Course Team
by Mick Bromilow

The Open University

The Open University, Walton Hall, Milton Keynes.

First published 1982. Reprinted 1983, 1985, 1989, 1992, 1994, 1996.

Designed by the Graphic Design Group of the Open University.

ISBN 0 335 14047 5

Printed and bound in the United Kingdom by Staples Printers Rochester Limited, Neptune Close, Medway City Estate, Frindsbury, Rochester, Kent ME2 4LT.

This text forms part of the correspondence element of an Open University Second Level Course.

For general availability of supporting material referred to in this text, please write to: Open University Educational Enterprises Limited, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, Great Britain.

Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

1.7

Contents

Introduction

Many methods of applied mathematics are based on the use of polynomial approximations, and in this unit we shall consider how some of these methods are derived and used to solve practical problems. In the next unit we shall make use of many of these results when solving differential equations numerically.

We consider two ways in which we might use a polynomial to approximate a function $f(x)$. If we know the value of the function and its first n derivatives at some point $x = \alpha$, say, as in Figure 1, then we can determine the **Taylor polynomial** of degree n about α which approximates $f(x)$ near the point $x = \alpha$.

In this unit the word *point* means a real number in the domain of the function.

In many problems the values of the derivatives are not known but the values of f are given at $n + 1$ points $x_0, x_1, x_2, \ldots, x_n$ as in Figure 2. In this case we can determine that polynomial of degree n which takes these values of f at $x_0, x_1, \ldots, x_n$. Such a polynomial is called an **interpolating polynomial**.

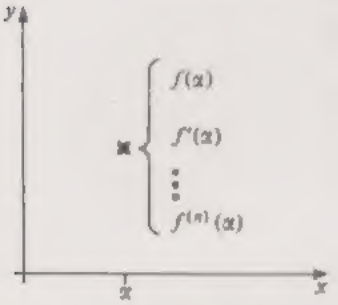


Figure 1. For Taylor polynomials the data is given at $x = \alpha$.

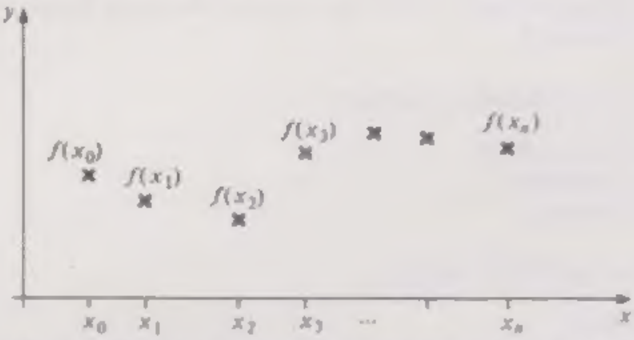


Figure 2. For interpolating polynomials the data is given at $n + 1$ points.

Taylor polynomials form the basis for two important methods in this unit: a method for finding roots of equations and a method for finding the approximate solutions of differential equations. Further, in the television programme we look at how Taylor polynomials can be used to derive approximate relationships between variables where it is impossible to find an exact relationship.

In the final section of the unit we use interpolating polynomials to derive methods for determining the approximate values of integrals, such as the trapezoidal method and Simpson's method.

Study guide

Section 1 of this unit revises the determination of Taylor polynomials and is central to the rest of the unit. Subsection 2.1 should be studied before Section 3, while Section 4 can be studied at any time after Section 1. Subsection 2.2, the tape subsection, can be studied at any time after Section 1, and is an important introduction to the next unit. Section 5 contains some extra exercises which you may attempt if you need more practice on the techniques introduced in this unit.

Before watching the television programme you should have studied Subsection 2.1, and Section 3 up to Subsection 3.3. The television programme concerns the study of freely hanging ropes and cables.

1 Basic theory

1.1 Polynomials

A **polynomial** is a function, $p(x)$ say, which can be expressed in the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where n is a non-negative integer and $a_0, a_1, \ldots, a_n$ are real numbers called the **coefficients** of the polynomial.

If n is the highest power of x for which the coefficient a_n is non-zero, $p(x)$ is said to be a **polynomial of degree n**. Hence the polynomial

$$1 + 7x + 4x^3$$

is a polynomial of degree 3.

Polynomials of low degree have acquired special names.

Degree	1	2	3	4	5
Name	linear	quadratic	cubic	quartic	quintic

These definitions are also given in Subsection 3.1 of the *Handbook*.

Thus the polynomial $1 + 7x + 4x^3$ is referred to as a *cubic polynomial* or simply a *cubic*.

The advantage of using polynomials is that they are very easy to manipulate algebraically. For example, if $p(x)$ is given by

$$p(x) = 1 + 3x + 4x^2 + x^3$$

and $q(x)$ is another polynomial given by

$$q(x) = 3 + 2x - x^2,$$

then we can, for example, add the two polynomials to get another polynomial

$$p(x) + q(x) = 4 + 5x + 3x^2 + x^3.$$

Polynomials are also very easy to differentiate or integrate. To differentiate

$$p(x) = 1 + 3x + 4x^2 + x^3,$$

we simply differentiate each term separately to obtain

$$\frac{dp}{dx}(x) = 3 + 8x + 3x^2,$$

while integrating $p(x)$ gives

$$\begin{aligned}\int p(x)\,dx &= \int (1 + 3x + 4x^2 + x^3)\,dx \\ &= C + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{1}{4}x^4\end{aligned}$$

where C is an arbitrary constant.

Example 1

The function $\tan x$ may be approximated near $x = 0$ by the polynomial

$$p(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5.$$

$p(x)$ is the 5th Taylor polynomial about 0 for $\tan x$. Taylor polynomials will be discussed in Subsection 1.2.

Use this approximation to estimate

$$\int_0^{0.5} \tan x\,dx.$$

Solution

$$\begin{aligned}\int_0^{0.5} \tan x\,dx &\simeq \int_0^{0.5} \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5\right) dx \\ &= \left[\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6\right]_0^{0.5} \\ &= 0.130\,556 \qquad \text{to six decimal places.}\end{aligned}$$

(The true solution, using the integration tables in Subsection 7.1 of the *Handbook*, gives

$$\int_0^{0.5} \tan x \, dx = \Big[-\log_e \cos x \Big]_0^{0.5}$$
$$= 0.130\,584 \quad \text{to six decimal places.}$$

Hence our approximation method gives a result which is correct to four decimal places.)

When evaluating a polynomial for a particular value of x, the most efficient method to use is called **nested multiplication**. This method will also usually reduce rounding errors in the calculations. To see how the method works we consider the polynomial

$$p(x) = 1 + 3x + 7x^2 + 4x^3.$$

If, for convenience, we reverse the order of the terms in the polynomial so that we write

$$p(x) = 4x^3 + 7x^2 + 3x + 1,$$

then the nested multiplication method can be used to evaluate $p(x)$ as

$$p(x) = \big((4x + 7)x + 3\big)x + 1.$$

The steps in the calculation are as follows.

(i) Enter 4 into the calculator.

(ii) Compute $4x + 7$.

(iii) Compute $(4x + 7) \times x + 3$.

(iv) Compute $\big((4x + 7)x + 3\big) \times x + 1$.

On my calculator the sequence of operations, given that x is stored in the memory and can be obtained using the RCL key, is

$\boxed{4}\ \boxed{\times}\ \boxed{\text{RCL}}\ \boxed{+}\ \boxed{7}\ \boxed{=}$
$\boxed{\times}\ \boxed{\text{RCL}}\ \boxed{+}\ \boxed{3}\ \boxed{=}$
$\boxed{\times}\ \boxed{\text{RCL}}\ \boxed{+}\ \boxed{1}\ \boxed{=}$

This sequence of operations can be expressed using a recurrence relation as

$$u_0 = 4$$
$$u_1 = xu_0 + 7$$
$$u_2 = xu_1 + 3$$
$$u_3 = xu_2 + 1,$$

and u_3 gives the value of the polynomial at x.

Example 2

Evaluate $p(x) = 4x^3 + 7x^2 + 3x + 1$ at $x = 1.27$.

Solution

Using the recurrence relation, we have

$$u_0 = 4$$
$$u_1 = 1.27u_0 + 7 = 12.08$$
$$u_2 = 1.27u_1 + 3 = 18.3416$$
$$u_3 = 1.27u_2 + 1 = 24.293\,832.$$

Notice that this method of evaluating the cubic requires only 3 multiplications and 3 additions.

The following procedure box gives the recurrence relation for evaluating a polynomial of degree n.

Nested multiplication

To evaluate

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

at a particular value of x, the nested multiplication method uses the recurrence relation

$$u_{r+1} = xu_r + a_{n-r-1}$$

where $u_0 = a_n$
and u_n gives the value of $p(x)$.

Exercise 1

This exercise concerns the polynomial

$$p(x) = 2x^4 + 3x^3 + x^2 - 7x + 2.$$

(i) What is the derivative of p?

(ii) What is the integral of p?

(iii) Evaluate $p(x)$ for $x = 0.29$ using nested multiplication.

(iv) Evaluate the derivative of p at $x = 0.77$.

[*Solution on p. 50*]

1.2 Taylor polynomials

There are many functions such as $\sin x$, e^x, $\log_e x$ and so on which we have to evaluate for a given value of x when carrying out computer or calculator computations. The traditional method, which you may have used, was to look up the required value of $\sin x$ in a table of sines. It is however not feasible for computers or calculators to store tables of numbers, since it would use up a relatively large amount of computer time to locate the required value. What happens in practice is that we approximate functions like the sine function by polynomials, so that for a given value of x we approximate $\sin x$ by the value of the polynomial.

There are several types of polynomial approximation, but traditionally the most popular are the Taylor polynomials. Suppose that we want to determine a polynomial $p(x)$ of degree n which approximates some given function $f(x)$ near $x = 0$. We write down $p(x)$ as

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

To derive the Taylor polynomial, we require p to be such that

$$\begin{aligned} p(0) &= f(0), \\ p'(0) &= f'(0) \\ p''(0) &= f''(0), \\ &\vdots \\ p^{(n)}(0) &= f^{(n)}(0). \end{aligned}$$

This derivation is also given in the Taylor Series units of *M101* and *MS283*.

This uniquely determines p, since

$$\begin{aligned} p(0) &= a_0, && \text{so that } a_0 = f(0); \\ p'(0) &= a_1, && \text{so that } a_1 = f'(0); \\ &\vdots \\ p^{(k)}(0) &= k!\,a_k, && \text{so that } a_k = \frac{f^{(k)}(0)}{k!}. \end{aligned}$$

Hence

$$p(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0). \tag{1}$$

This polynomial will normally give very good approximations to f near $x = 0$. (However, there are some functions, such as $|x|$, which cannot be approximated in this way since their derivatives do not exist at $x = 0$.) The above polynomial is known as the **nth Taylor polynomial about 0** for the function f.

You have already used one Taylor polynomial in Section 3 of *Unit 7*. See also Section 4 of the *Preparatory Booklet*, in which the alternative name nth Taylor approximation is used.

Here are two examples of Taylor polynomials.

Example 3

Determine the 3rd Taylor polynomial about 0 for the function

$$f(x) = e^x.$$

Solution

Differentiating this function is easy, since

$$\frac{d}{dx}e^x = e^x,$$

so that

$$f'(x) = e^x, \qquad f''(x) = e^x \qquad \text{and} \qquad f'''(x) = e^x.$$

At $x = 0$, $e^x = 1$, so that

$$f(0) = f'(0) = f''(0) = f'''(0) = 1.$$

Hence the 3rd Taylor polynomial about 0 approximating $f(x) = e^x$ is obtained by substituting into Equation (1), giving

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

In Figure 1 we have plotted the graphs of e^x and $p(x)$. Note that the two graphs are indistinguishable near $x = 0$.

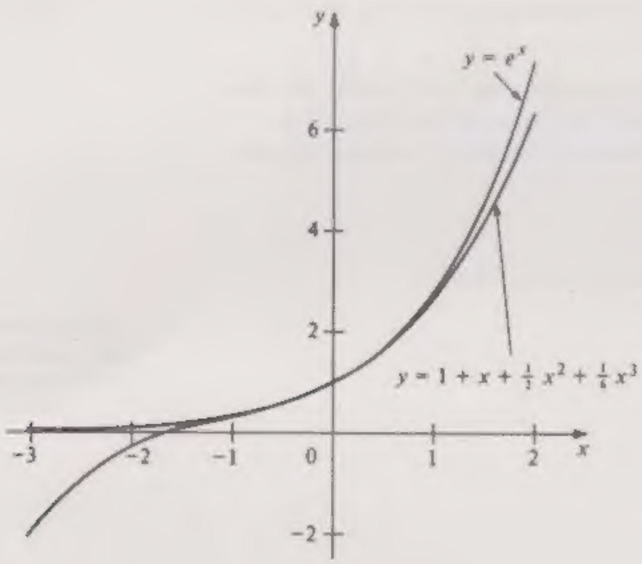


Figure 1. Graphs of e^x and $p(x) = 1 + x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3$.

Example 4

Consider the function

$$f(x) = (1 + x)^n,$$

where n is a positive integer. This is a polynomial of degree n and we would normally use the Binomial Theorem to obtain the expansion. However, we shall obtain the same polynomial expansion by looking at Taylor polynomials.

The Binomial Theorem is given in *M101* and *MS283*, and in Section 1 of the *Preparatory Booklet*.

The derivatives of $f(x)$ are

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} \\ f''(x) &= n(n-1)(1 + x)^{n-2} \\ &\vdots \\ f^{(n)}(x) &= n! \end{aligned}$$

All higher derivatives are zero.

At $x = 0$, $(1 + x)^k = 1$ for any value of k, so that

$$f(0) = 1, \;\; f'(0) = n, \;\; f''(0) = n(n-1), \;\; \ldots, \;\; f^{(n)}(0) = n!$$

Hence, using Equation (1), we have the nth Taylor polynomial about 0 for $f(x) = (1+x)^n$ as

$$p(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n.$$

Here $p(x)$ is exact because $(1+x)^n$ is a polynomial of degree n, and we can write

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.$$

In an exercise at the end of this subsection you will be asked to determine a Taylor polynomial approximation for $(1+x)^{1/2}$.

Very frequently we need approximations near some point $x = \alpha$ where α is not necessarily equal to zero. Suppose that we look for a polynomial of the form

$$p(x) = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + \cdots + a_n(x-\alpha)^n$$

such that

$$\begin{aligned} p(\alpha) &= f(\alpha) \\ p'(\alpha) &= f'(\alpha) \\ &\vdots \\ p^{(n)}(\alpha) &= f^{(n)}(\alpha). \end{aligned}$$

Such a polynomial is called the **nth Taylor polynomial about α** for the function f. Again it is not difficult to deduce that the coefficients of this polynomial are

$$a_k = \frac{f^{(k)}(\alpha)}{k!}, \qquad k = 0, 1, \ldots, n.$$

Thus we may define these Taylor polynomials as follows.

> The **nth Taylor polynomial about α** approximating the function $f(x)$ is defined by
>
> $$p(x) = f(\alpha) + (x-\alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2!}f''(\alpha) + \cdots + \frac{(x-\alpha)^n}{n!}f^{(n)}(\alpha).$$

To evaluate polynomials of the form

$$p(x) = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + \cdots + a_n(x-\alpha)^n,$$

the nested multiplication procedure can be adapted as

$$u_{r+1} = (x-\alpha)u_r + a_{n-r-1}$$

where $u_0 = a_n$ and u_n gives the value of $p(x)$. Thus polynomials in this form are as easy to evaluate as are ordinary polynomials.

When evaluating them, do *not* multiply out $(x-\alpha)^2$ as $x^2 - 2\alpha x + \alpha^2$, etc.

Example 5

Determine the 3rd Taylor polynomial about 1 approximating the function

$$f(x) = e^x.$$

Evaluate this polynomial at $x = 1.55$.

Solution

For $n = 0, 1, 2, \ldots$ we have $f^{(n)}(x) = e^x$ and $f^{(n)}(1) = e$. Thus,

$$\begin{aligned} p(x) &= e + (x-1)e + \tfrac{1}{2}(x-1)^2e + \tfrac{1}{6}(x-1)^3e \\ &= e\{1 + (x-1) + \tfrac{1}{2}(x-1)^2 + \tfrac{1}{6}(x-1)^3\}. \end{aligned}$$

This approximation is compared with e^x in Figure 2 (overleaf). In this case the graph of the polynomial is indistinguishable from that of e^x near $x = 1$.

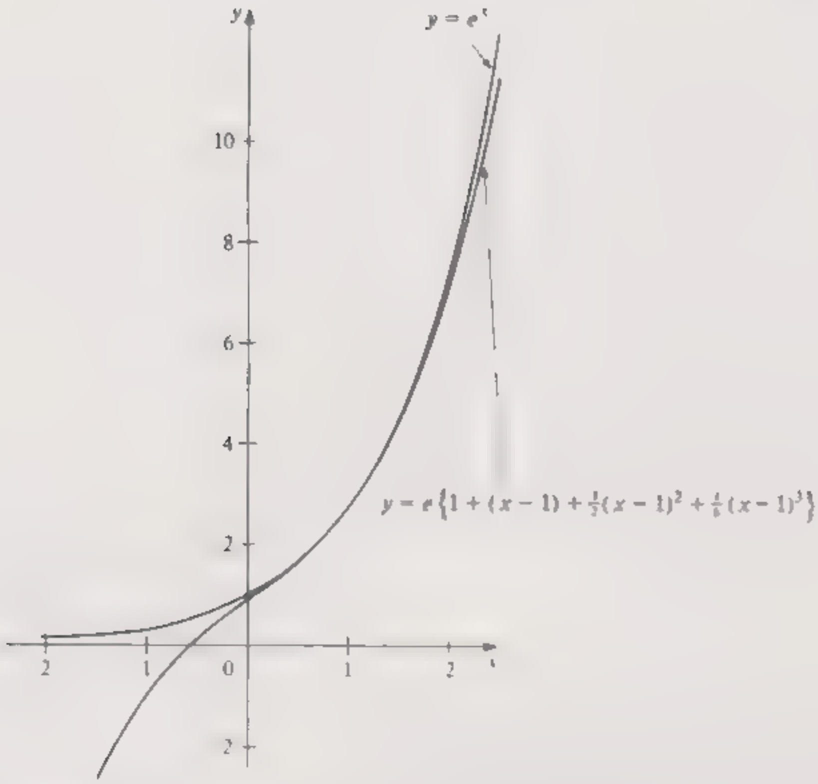


Figure 2. Graphs of $p(x) = e\{1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3\}$ and e^x.

To evaluate $p(x)$ at $x = 1.55$, compute $(x - 1) = 0.55$. Using nested multiplication to evaluate $1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x+1)^3$, we have

$$u_0 = 0.166\,666\,67$$
$$u_1 = 0.55u_0 + 0.5 = 0.591\,666\,67$$
$$u_2 = 0.55u_1 + 1 = 1.325\,416\,7$$
$$u_3 = 0.55u_2 + 1 = 1.728\,979\,2$$

On my calculator I stored 0.55 in the memory using the RCL key to recall it. Here is my key sequence

$\boxed{6}$ $\boxed{1/x}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{+}$ $\boxed{.}$ $\boxed{5}$ $\boxed{=}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{+}$ $\boxed{1}$ $\boxed{=}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{+}$ $\boxed{1}$ $\boxed{=}$

Hence $p(1.55) = eu_3 = 4.699\,85$ to five decimal places. This is not a bad approximation to the correct answer, which is 4.711 47 to five decimal places.

Exercise 2

Determine the 2nd Taylor polynomial about 0 approximating $f(x) = (1 + x)^{1/2}$. Evaluate this polynomial at $x = 0.44$ and compare your answer with the true solution.

Exercise 3

Show that the nth Taylor polynomial about 0 approximating $f(x) = \log_e(1 + x)$ is

$$p(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots + \frac{(-1)^{n+1}x^n}{n}.$$

[*Solutions to Exercises 2 and 3 on p. 50*].

1.3 Error bounds for Taylor polynomials

Wherever we use these approximations it is wise to obtain some estimate of the error in the approximation at a given value of x. That is, we would like to know something about the magnitude of the **error function** $\varepsilon(x)$ defined by

$$\varepsilon(x) = p(x) - f(x). \qquad (2)$$

For example, the error function for the third Taylor polynomial about 0 approximating e^x, given in Example 3, is

$$\varepsilon(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - e^x.$$

$\varepsilon(x)$ has been plotted in Figure 3.

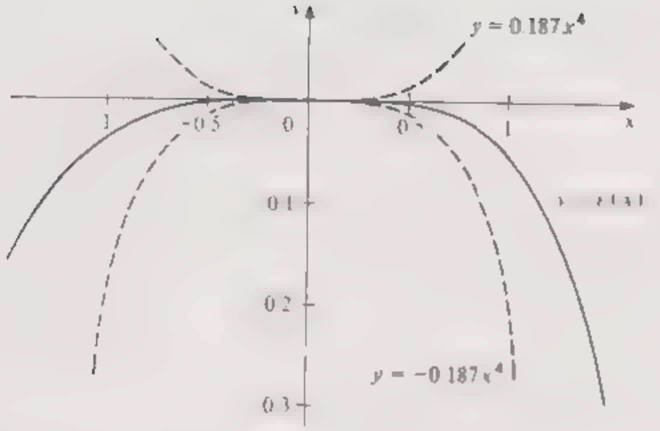


Figure 3. Graph of $\varepsilon(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - e^x$.

For comparison, the graphs of the functions $\pm 0.187x^4$ are also shown. (The reason for choosing these particular functions will become clear when you read Example 6 on pp. 12–13.)

The behaviour of $\varepsilon(x)$ near $x = 0$ in the above example should be expected because we have required that the first three derivatives of $p(x)$ and e^x should agree at $x = 0$. Hence $\varepsilon(0) = \varepsilon'(0) = \varepsilon''(0) = \varepsilon'''(0) = 0$, and so the first three Taylor polynomial approximations to $\varepsilon(x)$ are all zero.

Suppose now that we have obtained the nth Taylor polynomial $p(x)$ about α approximating some given function $f(x)$, and we wish to use this polynomial p to approximate f at a particular value of x. We need some idea of what the magnitude of the error is at this value of x. The theorem that follows gives a method of doing this. It is usually stated not in terms of the error function $\varepsilon(x)$ but in terms of a function called the **remainder**, defined as the function $r(x)$ in the formula

$$f(x) = p(x) + r(x).$$

Comparing with (2), we see that the remainder and the error function are related by

$$r(x) = -\varepsilon(x). \tag{3}$$

Taylor's Theorem

Let f and its first $n + 1$ derivatives exist and be continuous at all points between and including α and x. Then $f(x)$ can be expressed as

$$f(x) = f(\alpha) + (x - \alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2!}f''(\alpha) + \cdots$$

$$\cdots + \frac{(x-\alpha)^n}{n!}f^{(n)}(\alpha) + r(x) \tag{4}$$

where $r(x)$, the **remainder**, satisfies

$$r(x) = \frac{(x-\alpha)^{n+1}}{(n+1)!}f^{(n+1)}(c_x)$$

for some number c_x between α and x.

The requirement that the $n + 1$ derivatives of $f(x)$ should exist excludes functions such as $|x|$ or $x^{1/3}$ over an interval which includes 0.

I have used c_x here rather than c to indicate that c_x depends on the value of x.

The expression (4) is often called the **nth Taylor expansion** of $f(x)$ about α **with remainder**. The infinite sum

$$f(x) = f(\alpha) + (x - \alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2!}f''(\alpha) + \cdots + \frac{(x-\alpha)^n}{n!}f^{(n)}(\alpha) + \cdots \tag{5}$$

is called the **Taylor expansion** (or **Taylor series**) of $f(x)$ about α. Its significance is that, if we know for some particular value of x that the expression for the remainder,

$$r(x) = \frac{(x-\alpha)^{n+1}}{(n+1)!} f^{(n+1)}(c_x),$$

approaches zero as n becomes large, then we can conclude that (for that value of x) the sum of the terms in the Taylor series converges to $f(x)$. Although we will not be going into the question of convergence, the use of the Taylor series expression for a function is so widespread in applied mathematics that it is important to remember expression (5).

Rather than prove Taylor's Theorem I will just show what happens in the particular case when $n = 0$. In this case Taylor's Theorem states that if f and f' exist and are continuous for all points between and including α and x, then

$$f(x) = f(\alpha) + (x-\alpha) f'(c_x) \tag{6}$$

where c_x lies between α and x.

Rearranging Equation (6) gives

$$f'(c_x) = \frac{f(x) - f(\alpha)}{x - \alpha}.$$

Graphically this can be interpreted as saying that between α and x there must be a point c_x for which the slope of the graph $(f'(c_x))$ is the same as the slope of the (dashed) line joining the points $(\alpha, f(\alpha))$ and $(x, f(x))$ in Figure 4.

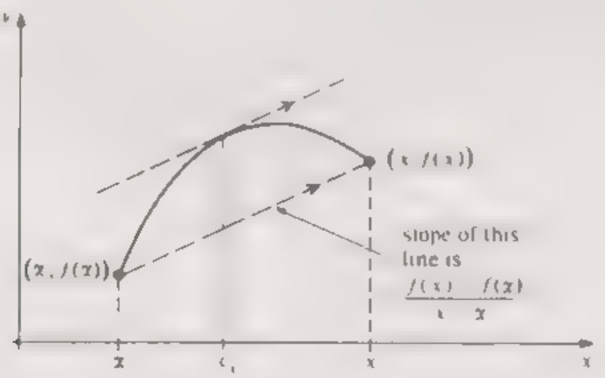


Figure 4. Graph illustrating Taylor's Theorem for $n = 0$.

This special case with $n = 0$ has an alternative name. It is often called the **Mean Value Theorem** for derivatives.

Since the remainder and the error are equal and opposite, we can use Taylor's Theorem to estimate the error function $\varepsilon(x)$. The following example shows how this can be done.

Example 6

(i) Verify that for all c satisfying $0 \leqslant c \leqslant 1.5$ we have $|e^c| \leqslant e^{1.5} = 4.4817\ldots$.

(ii) Deduce that for all x satisfying $0 \leqslant x \leqslant 1.5$ the error of the 3rd Taylor polynomial about 0 for e^x satisfies

$$|\varepsilon(x)| \leqslant Ax^4, \qquad \text{where } A = \frac{1}{24}e^{1.5}$$

Solution

(i) Since e^x is always positive we have $|e^c| = e^c$, and since $c \leqslant 1.5$ and the function is increasing we have $e^c \leqslant e^{1.5} = 4.4817\ldots$.

It follows that $|e^c| \leqslant 4.4817\ldots$ as required.

(ii) Taylor's Theorem, with $f(x) = e^x$, gives

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + r(x), \text{ where}$$
$$r(x) = \tfrac{1}{24}x^4 f^{(4)}(c_x)$$
$$= \tfrac{1}{24}x^4 \exp(c_x). \qquad (7)$$

Remember that exp (...) means e to the power of ... .

Since $0 \leqslant x \leqslant 1.5$, it follows that $0 \leqslant c_x \leqslant 1.5$ and hence, by part (i), that $|\exp(c_x)| \leqslant e^{1.5}$.

Using (7) we deduce that

$$|r(x)| \leqslant \tfrac{1}{24}x^4 e^{1.5}$$
$$= Ax^4, \qquad \text{where } A = \tfrac{1}{24}e^{1.5} \simeq 0.187.$$

Since $\varepsilon(x) = -r(x)$ (Equation (3)), we have $|\varepsilon(x)| = |r(x)|$, and hence

$$|\varepsilon(x)| \leqslant Ax^4$$

as required.

This result can also be written

$$-Ax^4 \leqslant \varepsilon(x) \leqslant Ax^4.$$

Figure 3 (page 11) shows that these inequalities are indeed satisfied for $0 \leqslant x \leqslant 1.5$, as the theorem states.

The procedure for determining the error bound used in Example 6 is formalized in the following box.

Error bounds for Taylor polynomials

1. Let $p(x)$ be the nth Taylor polynomial about α approximating a given function $f(x)$ which satisfies the conditions of Taylor's Theorem. We want to find a bound on the error function

$$\varepsilon(x) = p(x) - f(x)$$

for all x satisfying $a \leqslant x \leqslant b$, where a, b are given numbers such that $a \leqslant \alpha$ and $\alpha \leqslant b$.

2. Determine a value for M such that

$$|f^{(n+1)}(c)| \leqslant M$$

for all c satisfying $a \leqslant c \leqslant b$.

3. Taylor's Theorem gives

$$\varepsilon(x) = \frac{(x-\alpha)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$$

where c_x lies between α and x, and it follows since $a \leqslant c_x \leqslant b$, that

$$|\varepsilon(x)| \leqslant \frac{|x-\alpha|^{n+1}}{(n+1)!} M.$$

This is the required error bound.

Exercise 4

(i) Verify that for $0 \leqslant c \leqslant 0.5$ we have $|1 + c| \leqslant 1.5$.

(ii) Compute an error bound for the 3rd Taylor polynomial about 0 approximating $(1 + x)^5$, valid for all x satisfying $0 \leqslant x \leqslant 0.5$. (Note that you are *not* asked to compute the 3rd Taylor polynomial itself.)

Exercise 5

(i) Verify that $|\sin c| \leqslant 1$ for all c.

(ii) Determine the 5th Taylor polynomial about 0 approximating $\sin x$. Compute an error bound for this approximation at $x = 1$, using Taylor's Theorem.

[*Solutions to Exercises 4 and 5 on p. 50*]

Summary of Section 1

1. A **polynomial** is a function, $p(x)$ say, which can be expressed in the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where n is a non-negative integer and $a_0, a_1, \ldots, a_n$ are real numbers called the **coefficients** of the polynomial. If a_n is non-zero, p is called a **polynomial of degree** n.

2. To evaluate $p(x)$, written as

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

for a particular value of x, the **nested multiplication** method uses the recurrence relation

$$u_{r+1} = xu_r + a_{n-r-1}$$

where $u_0 = a_n$
and u_n gives the value of $p(x)$.

If $p(x)$ is of the form

$$p(x) = a_n(x-\alpha)^n + a_{n-1}(x-\alpha)^{n-1} + \cdots + a_1(x-\alpha) + a_0$$

then the recurrence relation is

$$u_{r+1} = (x-\alpha)u_r + a_{n-r-1}$$

where, as before, $u_0 = a_n$ and u_n gives the value of $p(x)$.

3. The **nth Taylor polynomial about** α approximating f is given by

$$p(x) = f(\alpha) + (x-\alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2!}f''(\alpha) + \cdots + \frac{(x-\alpha)^n}{n!}f^{(n)}(\alpha).$$

4. **Taylor's Theorem.** Let $f(x)$ and its first $n+1$ derivatives exist and be continuous at all points between and including α and x. Then $f(x)$ can be expressed as

$$f(x) = f(\alpha) + (x-\alpha)f'(\alpha) + \frac{(x-\alpha)^2}{2!}f''(\alpha) + \cdots + \frac{(x-\alpha)^n}{n!}f^{(n)}(\alpha) + \frac{(x-\alpha)^{n+1}}{(n+1)!}f^{(n+1)}(c_x),$$

where c_x lies between α and x and the expression $\frac{(x-\alpha)^{n+1}}{(n+1)!}f^{(n+1)}(c_x)$ is the **remainder**.

Hence the **error function** $\varepsilon(x)$ can be written as

$$\begin{aligned}\varepsilon(x) &= p(x) - f(x)\\ &= -\frac{(x-\alpha)^{n+1}}{(n+1)!}f^{(n+1)}(c_x).\end{aligned}$$

If a number M can be found such that

$$|f^{(n+1)}(c)| \leqslant M$$

for all c satisfying $a \leqslant c \leqslant b$, then it follows that

$$|\varepsilon(x)| \leqslant \frac{|x-\alpha|^{n+1}}{(n+1)!}M.$$

This is an upper bound on the error.

2 Two applications of Taylor polynomials

2.1 Roots of equations: the Newton-Raphson method

In the course of solving mathematical problems it is often necessary to find the values of x for which the equation

$$f(x) = 0$$

is satisfied, where f is some known function. I shall refer to them as the **roots** of the equation $f(x) = 0$.

They can also be called the *solutions* of the equation. You have already met these words, used interchangeably, in *Unit 5, Complex numbers.*

One way of determining the roots of an equation is to sketch the graph of

$$y = f(x)$$

and to determine those values x for which the curve cuts the x-axis.

For example, to find the roots of the equation

$$f(x) = 2x^3 - 5x^2 - 5x + 10 = 0$$

we could sketch the graph as in Figure 1.

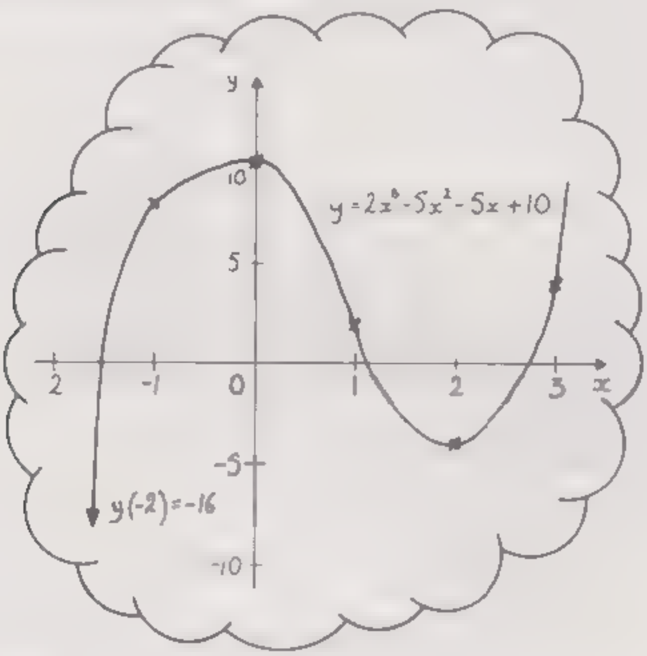


Figure 1. Sketch of $y = 2x^3 - 5x^2 - 5x + 10$.

Curve sketching is reviewed in Subsection 5.3 of the *Handbook*.

Looking at Figure 1 we can see that there are three values of x at which the graph cuts the x-axis. These are approximately -1.5, 1.2 and 2.8.

In very rare cases the graph may only touch the x-axis but not cross it, so that the graph may be as in Figure 2.

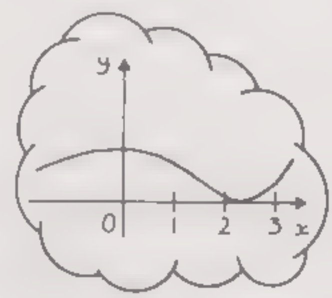


Figure 2. Graph of a function which touches but does not cross the x-axis

It can be difficult to determine roots in this instance, since with the naked eye we cannot detect whether there is one root, two roots or no roots as in the following close-ups.

(a) one root (b) two roots (c) no roots

Figure 3. Possible close-up views of the graph in Figure 2.

However in this unit we will assume that the graph is of the type given in Figure 1.

Another method of finding roots, which you have met before, is the formula method for quadratic equations such as

See Subsection 3.1 of the *Handbook*.

$$x^2 - 13x + 7 = 0.$$

Here are some equations for which the roots are not so easy to compute.

(i) $x - 2\sin x = 0$

(ii) $x^3 + 13.5x^2 + 40x + 16.67 = 0$

(iii) $xe^{-x} - 0.1 = 0$

What we want is a method which could be used to solve any of these problems. The method we will use is an application of Taylor polynomials and is called the **Newton-Raphson method**.

The Newton-Raphson method

This method determines the roots of an equation using a recurrence relation. We begin by supposing that there is a root of the equation $f(x) = 0$ near some point x_0. Now the first Taylor polynomial about x_0 approximating $f(x)$ is given by

The Newton-Raphson method was introduced in the Taylor Series units of *M101* and *MS283*.

$$p(x) = f(x_0) + (x - x_0)f'(x_0). \qquad (1)$$

This approximation is often called the **tangent approximation** to $f(x)$ at x_0 because it is the equation of a straight line which is tangential to the curve $y = f(x)$ at x_0, as in Figure 4.

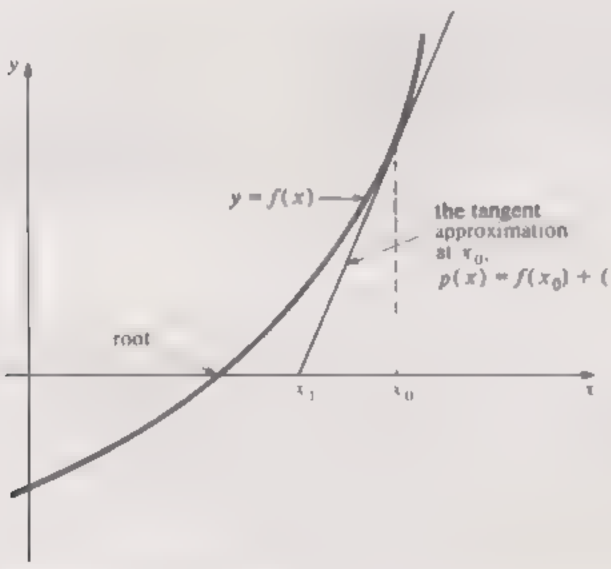


Figure 4. The tangent approximation to $f(x)$ at x_0.

In Figure 4, x_0 is not a particularly good approximation to the root of $f(x) = 0$, but we can use the tangent approximation at x_0 to improve this. The graph of the tangent approximation cuts the x-axis at the point x_1 such that

$$p(x_1) = 0.$$

Since p is an approximation to f, the root, x_1, of the equation $p(x) = 0$ is likely to be a close approximation to the root of $f(x) = 0$.

To calculate x_1 we use Equation (1), which tells us that $p(x_1) = 0$ when

$$f(x_0) + (x_1 - x_0)f'(x_0) = 0.$$

Hence rearranging gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \qquad (2)$$

A slightly different derivation of Equation (2) will be given in the television programme.

Remember to use nested multiplication to evaluate $f(x)$.

Example 1

From Figure 1 we know that there is a root of the equation

$$f(x) = 2x^3 - 5x^2 - 5x + 10 = 0$$

near $x_0 = -1.5$.

Now $f(-1.5) = -0.5$
and $f'(x) = 6x^2 - 10x - 5$, giving

$$f'(-1.5) = 23.5.$$

Using Equation (2) with $x_0 = -1.5$ gives

$$x_1 = -1.5 - \frac{f(-1.5)}{f'(-1.5)} = -1.5 + \frac{0.5}{23.5} = -1.478\,723\,4.$$

We can see that x_1 is probably a much better approximation to the root than x_0, since $f(-1.478\,723\,4) = -0.006\,318\,39$ and this is considerably smaller in magnitude than $f(-1.5) = -0.5$.

We need not stop here. We can compute the tangent approximation to f at x_1 to obtain an even better approximation, x_2, to the root of $f(x) = 0$, as in Figure 5.

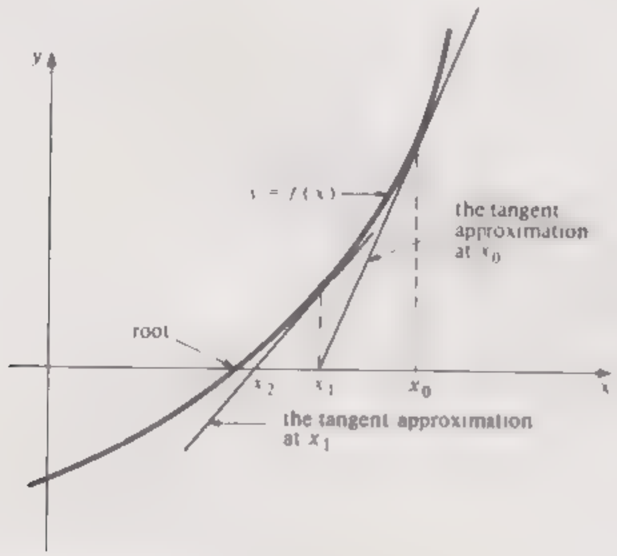


Figure 5. The tangent approximation at x_1.

The equation for x_2 can be deduced from Equation (2) as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Using this value of x_2 we can compute x_3 using another tangent approximation at x_2, and so on. Each term in the sequence $x_0, x_1, x_2, \ldots$ is computed using the recurrence relation

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}. \tag{3}$$

Example 2

In Example 1 we looked for a root of the equation

$$2x^3 - 5x^2 - 5x + 10 = 0$$

near $x_0 = -1.5$ and computed $x_1 = -1.478\,723\,4$. Continuing with this example, we build up the following table using Equation (3). (You may get slightly different results with your calculator.)

r	x_r	$f(x_r) = 2x_r^3 - 5x_r^2 - 5x_r + 10$	$f'(x_r) = 6x_r^2 - 10x_r - 5$	$x_{r+1} = x_r - \dfrac{f(x_r)}{f'(x_r)}$
0	-1.5	-0.5	23.5	$-1.478\,723\,4$
1	$-1.478\,723\,4$	$-0.006\,318\,39$	22.906 971	$-1.478\,447\,6$
2	$-1.478\,447\,6$	$-0.000\,001\,06$	22.899 319	$-1.478\,447\,5$
3	$-1.478\,447\,5$	-2×10^{-8}		

After only three iterations $f(x_3)$ is so small that we can conclude that $x_3 = -1.478\,447\,5$ is a very good approximation to the root

The precise value obtained for $f(x_3)$ depends quite strongly on which calculator is used and how the calculation is done, but is very small whatever method is used

There are two possible criteria for stopping the sequence.

(i) If $f(x_r)$ is very small, then x_r is assumed to be very close to the root

(ii) If $x_{r+1} \simeq x_r$, then Equation (3) gives

$$x_r \simeq x_r - \frac{f(x_r)}{f'(x_r)}, \text{ which is equivalent to } \frac{f(x_r)}{f'(x_r)} \simeq 0.$$

Hence provided that $f'(x_r)$ is not too large we can conclude that x_r is again a good approximation to the root.

If we apply these criteria to Example 2, we find respectively:

(i) $f(x_3) = -2 \times 10^{-8}$;

(ii) $x_2 = -1.478\,447\,6, \qquad x_3 = -1.478\,447\,5$;

and in either case we could conclude that there is no point in computing further terms in the sequence.

We summarize the Newton-Raphson method in the following procedure box

The Newton-Raphson method
Given an initial approximation x_0 close to the desired root of $f(x) = 0$.

Generate a sequence $x_0, x_1, x_2, \ldots$ of successively better approximations using the recurrence relation

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

This procedure can be terminated if

either: (i) $f(x_r)$ becomes sufficiently small

or: (ii) $x_{r+1} \simeq x_r$.

To begin using the Newton-Raphson method we need an initial approximation x_0. If we do not have any information about x_0 the simplest thing to do is to sketch a graph of $y = f(x)$, as we did for Example 1. The sketch need not be particularly accurate, and plotting half a dozen points will usually be sufficient to obtain crude approximations to the roots. In the following example, we suggest an alternative curve sketching method which sometimes can reduce the effort required to find a value for x_0 still further.

Example 3

Find all the roots of

$$x - 2\sin x = 0$$

using the Newton-Raphson method. Quote your answers to five significant figures

Solution

To use the Newton-Raphson method I need a value for x_0. To obtain this I could sketch a graph of

$$f(x) = x - 2\sin x$$

and look for the roots. However, it is easier in this case to draw graphs of $y = x$ and $y = 2\sin x$ and to look for points at which the two graphs intersect, since at these points we have $x = 2\sin x$.

We can deduce that there can be no roots if $|x| > 2$, since $-1 \leqslant \sin x \leqslant 1$, so that $-2 \leqslant 2\sin x \leqslant 2$ and the only possible roots of $x = 2\sin x$ are in the interval $-2 \leqslant x \leqslant 2$.

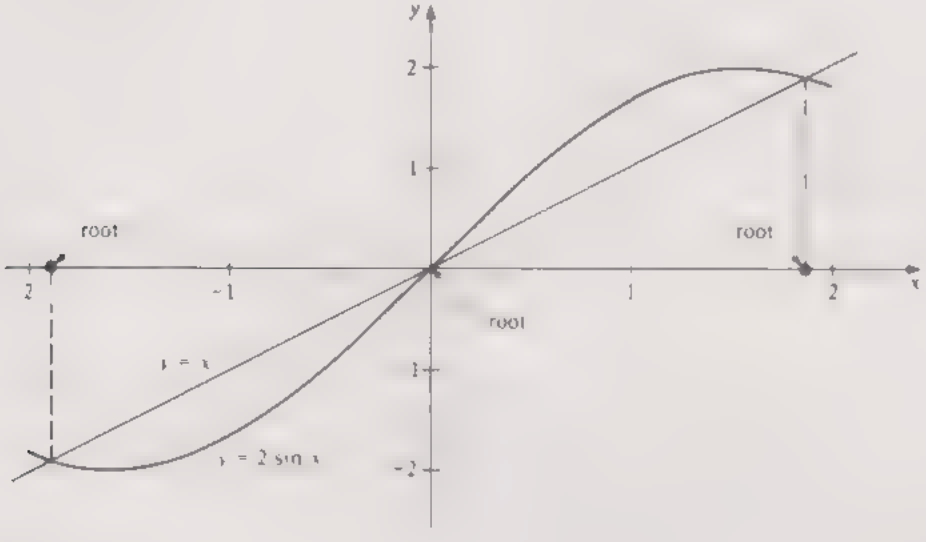


Figure 6. Graphs of $y = x$ and $y = 2\sin x$ for $-2 \leqslant x \leqslant 2$.

From the graphs in Figure 6 we can see that there are three roots. The first is at $x = 0$, while the other two roots are approximately -1.9 and 1.9.

Since we know that $x = 0$ is the exact value of a root, we do not need to use the Newton-Raphson method for this one.

The Newton-Raphson recurrence relation is

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}.$$

In this case, $f(x) = x - 2\sin x$ and $f'(x) = 1 - 2\cos x$. Hence the recurrence relation is

$$x_{r+1} = x_r - \frac{x_r - 2\sin x_r}{1 - 2\cos x_r}.$$

With $x_0 = 1.9$ I obtained the following sequence, using my calculator.

$$\begin{aligned} x_0 &= 1.9 \\ x_1 &= 1.895\,505\,9 \\ x_2 &= 1.895\,494\,3 \qquad (f(x_2) = -9 \times 10^{-10}) \\ x_3 &= 1.895\,494\,3 \qquad (f(x_3) = -4 \times 10^{-10}) \end{aligned}$$

You may get slightly different results using your calculator.

After only three iterations the values of x_r agree to eight figures and $f(x_3)$ is exceedingly small.

Thus the root to five significant figures is 1.8955.

Similarly the third root is at -1.8955. Hence the three roots are 0 and ± 1.8955.

We know from basic trigonometry that $\sin(-x) = -\sin x$, so that if α is a root of $x = 2\sin x$ then so is $-\alpha$.

In the above example the sequence converged very rapidly to the required root. The question we might ask is whether this will always be so. In the following piece of theory we will explain this rapid convergence and also gain some insight into how accurately we need to determine x_0, the initial guess at the root.

The recurrence relation for the Newton-Raphson method given by Equation (3) can be rearranged as

$$0 = f(x_r) + (x_{r+1} - x_r)f'(x_r). \tag{4}$$

Let us compare Equation (4) with the first Taylor expansion about x_r with remainder (page 11). Assuming that $f(x)$ satisfies the conditions of Taylor's Theorem (with $n = 1$), we have

$$f(x) = f(x_r) + (x - x_r)f'(x_r) + \tfrac{1}{2}(x - x_r)^2 f''(c_x) \tag{5}$$

where c_x lies between x_r and x.

If ρ is a root of the equation $f(x) = 0$, so that

$$f(\rho) = 0,$$

then we can substitute ρ for x in Equation (5) to get

$$0 = f(x_r) + (\rho - x_r)f'(x_r) + \tfrac{1}{2}(\rho - x_r)^2 f''(c_\rho) \tag{6}$$

where c_ρ lies between x_r and ρ.

Subtracting Equation (6) from Equation (4) gives

$$0 = (x_{r+1} - \rho)f'(x_r) - \tfrac{1}{2}(\rho - x_r)^2 f''(c_\rho),$$

i.e.

$$x_{r+1} - \rho = \tfrac{1}{2}(\rho - x_r)^2 \frac{f''(c_\rho)}{f'(x_r)}. \tag{7}$$

Look at this last equation, (7), as this gives the secret of the success of the Newton-Raphson method. The left-hand side of the equation, $x_{r+1} - \rho$, is just the error in the $(r + 1)$th approximation x_{r+1} to the root ρ. Equation (7) relates this error to the error in x_r. It shows that the error in x_{r+1} is proportional to the square of the error in x_r; so if the error in x_r is fairly small, the error in x_{r+1} will be much smaller. As soon as we are reasonably close to the root, the errors diminish very rapidly.

In order to say more precisely when the method will converge, using Equation (7), we need to know some values for

$$\frac{f''(c_\rho)}{2f'(x_r)},$$

which are generally not available. However, we can say that provided the conditions of Taylor's Theorem are met and the equation $f'(x) = 0$ does not have a root near ρ, then the sequence of iterates is guaranteed to converge for a suitable starting value x_0. At the beginning of Subsection 2.1 I stated that we would assume that the function did not behave as in Figure 2. This figure shows a situation in which the equation $f'(x) = 0$ does have a root near ρ, so that the Newton-Raphson method would not work well.

Exercise 1

Find all three roots, to 5 significant figures, of the cubic equation

$$x^3 + 13.5x^2 + 40x + 16.67 = 0$$

using the Newton-Raphson method. Figure 7 shows a sketch of the graph.

[*Solution on p. 50*]

The results of this exercise will be used in *Unit 24, Normal modes*.

If you are short of time, just calculate one of the roots.

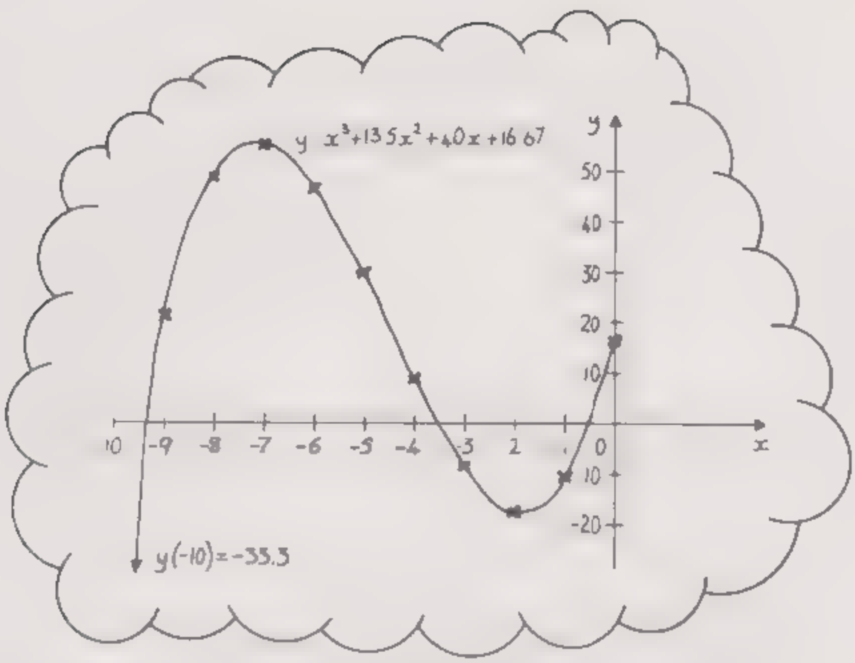


Figure 7. Sketch of $y = x^3 + 13.5x^2 + 40x + 16.67$.

Exercise 2

(i) Show that the roots of the equation

$$xe^{-x} - 0.1 = 0$$

are at the intersections of the graphs of $y = e^x$ and $y = 10x$, and hence find rough approximations to them.

(ii) Use the Newton-Raphson method to determine these roots to 6 decimal places

[*Solution on p. 51*]

2.2 Taylor series methods for differential equations (Tape Subsection)

The second application of Taylor polynomials that we look at is in the numerical solution of differential equations of the form

$$y' = m(x, y) \qquad \text{with } y(x_0) \text{ given}$$

where $m(x, y)$ is some given formula which may contain both x and y.

You saw many differential equations of this form in *Unit 2*.

For example, we may be asked to find an approximate solution to the differential equation

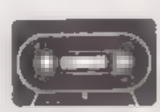
$$y' = x + y$$

for values of x such that $0 \leqslant x \leqslant 1$, given that $y = 0$ when $x = 0$. This is the example we shall use in the tape frames which follow.

(In this particular case we could compute an analytical solution using the integrating factor method, but we use this simple example here to illustrate the numerical method.)

The true solution is $y = e^x - x - 1$

You have already met one method, *Euler's method*, in *Unit 2* for solving such problems numerically. The methods we shall develop here are called **Taylor series methods.** In general Taylor series methods will give much more accurate results than Euler's method.



Start the tape now.

1

Taylor polynomials

Differential equation : $y' = x + y$ (1)

Initial condition : $y(0) = 0$

PROBLEM : Use a 2nd Taylor polynomial approximation for y about 0 to get an approximate value for y at $x = 0\cdot1$

$$p(x) = y(0) + xy'(0) + \tfrac{1}{2}x^2y''(0) \qquad (2)$$

DATA : $y(0) = \boxed{}$

$y'(0) = \boxed{}$

$y''(0) = \boxed{}$

$x = \boxed{0\cdot1}$

SOLUTION : $p(0\cdot1) = \boxed{}$

$= \boxed{}$

2

Computing the second derivative

The differential equation (1) is

$$y' = x + y$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(x + y)$$

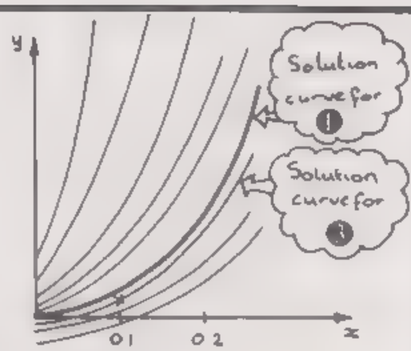
Thus $y'' = \boxed{}$ (3)

Hence $y''(0) = \boxed{}$

$= \boxed{}$

3

A new problem



Differential equation : $y' = x + y \quad (4)$

Initial condition : $y(0{\cdot}1) = 0{\cdot}005$

PROBLEM : Use a 2nd Taylor polynomial about 0 to get an approximate value for y at $x = 0{\cdot}2$

$$p(x) = y(0{\cdot}1) + (x - 0{\cdot}1)\,y'(0{\cdot}1) + \tfrac{1}{2}(x - 0{\cdot}1)^2 y''(0{\cdot}1) \qquad (5)$$

DATA : $y(0{\cdot}1) =$

$y'(0{\cdot}1) =$

$y''(0{\cdot}1) =$

$x =$

SOLUTION : $p(0{\cdot}2) =$

$=$

4

The general problem

Differential equation : $y' = m(x, y)$

Initial condition : $y(x_r) = \alpha$

PROBLEM : Use a 2nd Taylor polynomial about x_r to get an approximate value for y at $x = x_{r+1}$

$$p(x) = y(x_r) + (x - x_r)\,y'(x_r) + \tfrac{1}{2}(x - x_r)^2 y''(x_r) \qquad (6)$$

$$\text{where } y'(x_r) = m\big(x_r, y(x_r)\big) \qquad (6a)$$

$$\text{and } y''(x_r) = \frac{dm}{dx}\big(x_r, y(x_r)\big) \qquad (6b)$$

SOLUTION : $p(x_{r+1}) = y(x_r) + (x_{r+1} - x_r)\,y'(x_r) + \tfrac{1}{2}(x_{r+1} - x_r)^2 y''(x_r) \quad (7)$

5

Simplification

Write y_r for $y(x_r)$, y'_r for $y'(x_r)$ and y''_r for $y''(x_r)$

Write $h = x_{r+1} - x_r$ as the step size

Equation (7) becomes

$$p(x_{r+1}) = y_r + hy'_r + \tfrac{1}{2}h^2 y''_r \qquad (8)$$

where $y'_r = m(x_r, y_r)$

and $y''_r = \dfrac{dm}{dx}(x_r, y_r)$

6

Reworking the problem in Frame 1

Differential equation : $y' = x + y$

Initial condition : $y(0) = 0$

Taylor series method of order 2 with $h = 0{\cdot}1$

$$p(x_1) = y_0 + hy'_0 + \tfrac{1}{2}h^2 y''_0 \qquad (9)$$

where $y'_0 = m(x_0, y_0) = x_0 + y_0 = \boxed{}$ (9a)

and $y''_0 = \dfrac{dm}{dx}(x_0, y_0) = 1 + y'_0 = \boxed{}$ (9b)

$p(0{\cdot}1) = \boxed{}$

$= \boxed{}$

7

Notation for the approximate solution

Given y_0, we want values for $y_1, y_2, y_3, \cdots, y_n$

Approximations : Given $Y_0 = y_0$ we actually compute $Y_1, Y_2, Y_3, \cdots, Y_n$

The Taylor series method of order 2 uses the recurrence relation

$$Y_{r+1} = Y_r + hY'_r + \tfrac{1}{2}h^2 Y''_r \qquad (10)$$

where $Y'_r = m(x_r, Y_r)$

and $Y''_r = \dfrac{dm}{dx}(x_r, Y_r)$

8

A typical problem

Compute the approximate solution to the differential equation

$$y' = x + y,$$

with $y(0) = 0$, at $x = 0{\cdot}1, 0{\cdot}2, \cdots, 1$ using the Taylor series method of order 2.

r	x_r	Y_r	$Y_r' = x_r + Y_r$	$Y_r'' = 1 + Y_r'$	$Y_{r+1} = Y_r + hY_r' + \frac{1}{2}h^2 Y_r''$
0	0	0	0	1	0.005
1	0.1	0.005	0.105	1.105	0.021025
2	0.2	0.021025	______	______	______
3	0.3	0.04923263	______	______	______
4	0.4	0.09090205	0.49090205	1.49090205	0.14744676
5	0.5	0.14744676	0.64744676	1.64744676	0.22042866
6	0.6	0.22042866	0.82042866	1.82042866	0.31157366
7	0.7	0.31157366	1.01157366	2.01157366	0.42278889
8	0.8	0.42278889	1.22278889	2.22278889	0.55618171
9	0.9	0.55618171	1.45618171	2.45618171	0.71408079
10	1.0	0.71408079			

9

Taylor series method of order 3

Compute the approximate solution to the differential equation

$$y' = x + y,$$

with $y(0) = 0$, at $x = 0.1, 0.2, \dots, 1$, using the Taylor series method of order 3.

$$y'' = 1 + y'$$

$$y''' = \frac{d}{dx}(y'') = \boxed{\quad} \qquad (11)$$

r	x_r	Y_r	$Y_r' = x_r + Y_r$	$Y_r''' = Y_r'' = 1 + Y_r'$	$Y_{r+1} = Y_r + hY_r' + \frac{1}{2}h^2 Y_r'' + \frac{1}{6}h^3 Y_r'''$
0	0	0	0	1	0.005 166 67
1	0.1	0.005 166 67	0.105 166 67	1.105 166 67	0.021 393 35
2	0.2	0.021 393 35	______	______	______
3	0.3	0.049 843 22	______	______	______
4	0.4	0.091 801 73	0.491 801 73	1.491 801 73	0.148 689 54
5	0.5	0.148 689 54	0.648 689 54	1.648 689 54	0.222 076 72
6	0.6	0.222 076 72	0.822 076 72	1.822 076 72	0.313 698 45
7	0.7	0.313 698 45	1.013 698 45	2.013 698 45	0.425 472 4
8	0.8	0.425 472 4	1.225 472 4	2.225 472 4	0.559 517 91
9	0.9	0.559 517 91	1.459 517 91	2.459 517 91	0.718 177 21
10	1.0	0.718 177 21			

We summarize the procedure for the Taylor series method of order 2 in the following procedure box.

Taylor series method of order 2

1. You are given a differential equation

$$y' = m(x, y)$$

and a value for $y(x_0)$.

2. Differentiate the expression for $m(x, y)$ to obtain $\dfrac{dm}{dx}(x, y)$, remembering that y is a function of x.

3. To apply the Taylor series method of order 2, choose a step size h, and calculate $Y_1, Y_2, \ldots$ from

$$Y_{r+1} = Y_r + hY_r' + \tfrac{1}{2}h^2 Y_r''$$

where

$$Y_r' = m(x_r, Y_r)$$

$$Y_r'' = \frac{dm}{dx}(x_r, Y_r)$$

$$Y_0 = y(x_0) \text{ and } x_r = x_0 + rh.$$

4. Y_r is an approximation to y_r.

The notation $\dfrac{dm}{dx}(x_r, Y_r)$ means that we evaluate $\dfrac{dm}{dx}(x, y)$ at $x = x_r$ and $y = Y_r$.

The procedure for using the Taylor series method of order n can be stated, in a similar way, as follows.

Taylor series method of order n

1. You are given a differential equation

$$y' = m(x, y)$$

and a value for $y(x_0)$.

2. Differentiate the expression for $m(x,y)$ $n-1$ times to obtain expressions for $\dfrac{dm}{dx}(x,y), \dfrac{d^2m}{dx^2}(x,y), \ldots, \dfrac{d^{n-1}m}{dx^{n-1}}(x,y)$, remembering that y is a function of x.

3. To apply the Taylor series method of order n, choose a step size h, and calculate $Y_1, Y_2, \ldots$ from

$$Y_{r+1} = Y_r + hY_r' + \frac{h^2}{2!}Y_r'' + \cdots + \frac{h^n}{n!}Y_r^{(n)}$$

where

$$Y_r' = m(x_r, Y_r)$$

$$Y_r'' = \frac{dm}{dx}(x_r, Y_r)$$

$$\vdots$$

$$Y_r^{(n)} = \frac{d^{n-1}m}{dx^{n-1}}(x_r, Y_r)$$

$$Y_0 = y(x_0) \quad \text{and} \quad x_r = x_0 + rh.$$

4. Y_r is an approximation to y_r.

There is a decision to be made here whether to use a higher-order Taylor series method with a larger step size or a lower-order Taylor series method with smaller step size in order to achieve a required accuracy. Reducing the step size can reduce the error significantly, but requires correspondingly more computational effort to

obtain approximations over a range of x-values. Adding an extra term in the Taylor series would probably not involve significantly more work provided that the derivative were easy to obtain. We shall say more about such comparisons in the next unit.

One of the main drawbacks of the Taylor series methods is that they involve the use of higher derivatives which may be very complicated expressions. For example, you might like to convince yourself that the implementation of the Taylor series method of order 4 for the differential equation

$$y' = \sqrt{x^2 + y^2}$$

is by no means easy. (Do not spend more than a couple of minutes doing this.)

In addition the work involved in inputting these higher derivatives into a computer or programmable calculator may be prohibitive. Consequently in the next unit we will look at other methods, which do not require these higher derivatives.

Exercise 3

Write down the recurrence relation for the Taylor series method of order 1. Does this method have another name?

Exercise 4

Use the Taylor series method of order 2 to solve the differential equation

$$y' = 3y + \sin x \qquad \text{with } y(0) = 0.$$

Use $h = 0.2$ to compute approximate solutions at $x = 0.2$ and $x = 0.4$.

Exercise 5

Write down the recurrence relation and expressions for Y_r' (in terms of x_r and Y_r), Y_r'' (in terms of x_r, Y_r and Y_r') and Y_r''' (in terms of x_r, Y_r, Y_r' and Y_r'') required to use the Taylor series method of order 3 to solve the differential equation

$$y' = \sin y \qquad \text{with } y(0) = 1.$$

Taking $h = 0.1$, compute the approximate solution at $x = 0.1$.

[*Solutions to Exercises 3–5 on p. 51*]

Summary of Section 2

1. A root of the equation

$$f(x) = 0$$

is any value of x for which the equation is satisfied.

2. The **Newton-Raphson method**, given an initial approximation x_0 close to the root of

$$f(x) = 0,$$

can be used to generate a sequence $x_0, x_1, x_2, \ldots$ of successively better approximations to the root using the recurrence relation

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

The error in x_{r+1} is proportional to the square of the error in x_r, and so if the sequence converges it does so very rapidly.

3. The **Taylor series method of order 2** for solving first-order differential equations is given on page 27.

4. The **Taylor series method of order n** for solving first-order differential equations is given on page 27.

5. The selection of the appropriate order of the Taylor series method and the choice of step size are difficult to determine for a given differential equation. The drawback of the method is that higher derivatives may be complicated expressions which are tedious to work out and time-consuming to input to a computer or programmable calculator.

3 The catenary (Television Section)

To illustrate how some of the techniques of this unit are used in practice, we are going to look at a particular problem which arises out of the study of catenaries. A **catenary** is the shape taken by a freely hanging rope or cable fixed at both ends, as shown in Figure 1.

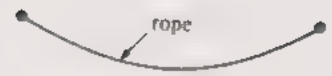


Figure 1. A catenary.

Here is a statement of the problem we are going to tackle. Take a length L of rope (where L is greater than 2 m) and suspend it between two points 2 m apart in the same horizontal plane. What is the displacement d of the middle of the rope below this plane? (L and d are shown in Figure 2.)

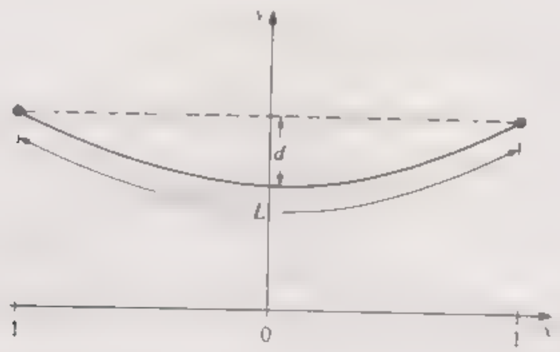


Figure 2. Television problem is to find d given L.

This problem has applications in the design of telephone cables or electricity cables, in that temperature changes cause the cables to increase or decrease in length.

It would be useful if we could find a formula expressing the displacement d in terms of the length L, because it would then be possible to determine d by substituting the given value of L in that formula. Unfortunately such a formula is impossible to find. However, we shall see that the Newton-Raphson method can be used to determine d for any given value of L. It is also possible to use Taylor polynomial approximations to determine an approximate relationship between d and L which gives good answers for small displacements. To tackle this problem we need two pieces of information:

(i) the equation expressing the catenary as a function of x;

(ii) a formula for the length of a curve.

These two topics are the subject of the next subsection, which you should read before watching the television programme.

3.1 Hyperbolic functions

The modelling of the catenary, which assumes that the mass per unit length is constant along the length of the rope, leads to a second-order differential equation. Unfortunately we do not have time to go into the details here. For a suitable choice of the x- and y-axes the equation of the catenary can be written as

They are given, for example, in *Differential Equations with Applications and Historical Notes* by G. F. Simmons, McGraw-Hill, 1972.

$$y = \frac{1}{a}\left(\frac{e^{ax}+e^{-ax}}{2}\right) \tag{1}$$

where a is a constant whose value is chosen to select the particular catenary we want from a whole family of catenaries. (For each length L of rope we will have a different value of a.) The suitable choice of axes (see Figure 3) is such that

$$y'(0) = 0$$

and $$y(0) = \frac{1}{a}.$$

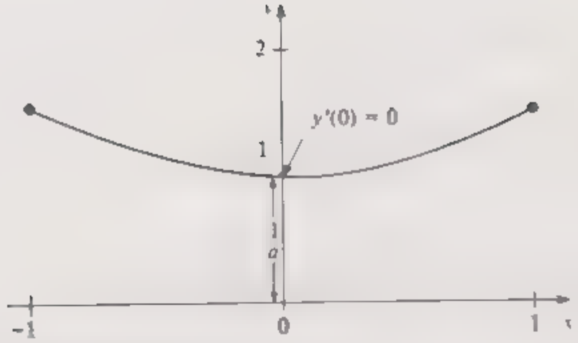


Figure 3. Axes for the catenary in the case $a = 1$.

The expression in brackets in Equation (1) appears so often that it is given a name. We define the **hyperbolic cosine** of the number ax as

$$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}.$$

The symbol cosh is pronounced as 'cosh'.

The corresponding **hyperbolic sine** is defined as

$$\sinh ax = \frac{e^{ax} - e^{-ax}}{2}.$$

The symbol sinh is pronounced as 'shine'.

The equation of the catenary can now be written as

$$y = \frac{1}{a}\cosh ax. \qquad (2)$$

The hyperbolic functions have some very interesting properties, which we will use in the television programme. These properties are very similar to the properties of ordinary sines and cosines. In fact, to distinguish between the two types of function we call the well-known sines and cosines **circular functions**. This name arises because any point on a unit circle can be defined in terms of the cosine and sine of the angle θ shown in Figure 4.

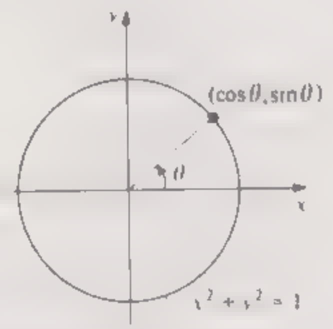


Figure 4. Any point on the unit circle can be expressed in terms of θ.

Since the equation of the unit circle is given by

$$x^2 + y^2 = 1,$$

the well-known equation

$$\cos^2\theta + \sin^2\theta = 1$$

shows that the point $(\cos\theta, \sin\theta)$ lies on this circle.

To see why we call cosh and sinh hyperbolic functions, consider the curve, called a **hyperbola**, whose equation is

$$x^2 - y^2 = 1. \qquad (3)$$

I shall show that the point whose coordinates are $(\cosh\theta, \sinh\theta)$ lies on the hyperbola, i.e. that it satisfies Equation (3). The number θ can take any value, but its geometical significance is less obvious than for the circle.

Now

$$\cosh\theta + \sinh\theta = \left(\frac{e^{\theta} + e^{-\theta}}{2}\right) + \left(\frac{e^{\theta} - e^{-\theta}}{2}\right) = e^{\theta}$$

and

$$\cosh\theta - \sinh\theta = \left(\frac{e^{\theta} + e^{-\theta}}{2}\right) - \left(\frac{e^{\theta} - e^{-\theta}}{2}\right) = e^{-\theta}$$

Thus

$$\begin{aligned}\cosh^2\theta - \sinh^2\theta &= (\cosh\theta + \sinh\theta)(\cosh\theta - \sinh\theta)\\ &= e^{\theta}e^{-\theta} = 1\end{aligned}$$

as required.

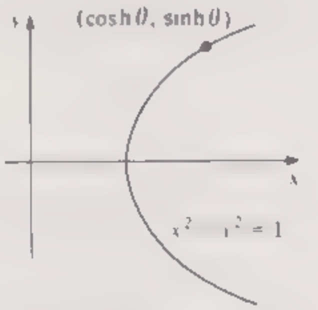


Figure 5. Any point on the hyperbola $x^2 - y^2 = 1$ can be expressed in terms of θ.

Note the similarity between the result

$$\cosh^2\theta - \sinh^2\theta = 1$$

for hyperbolic functions and the result

$$\cos^2\theta + \sin^2\theta = 1$$

for circular functions.

The following table lists some properties of hyperbolic functions together with the corresponding properties of circular functions. They can be derived using the definitions of $\cosh x$ and $\sinh x$ in terms of exponential functions. If you have time you might verify some of these properties. You are not expected to memorize these; they will be included in the *Handbook*.

Hyperbolic functions	*Circular functions*
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$
$\cosh^2 x - \sinh^2 x = 1$	$\cos^2 x + \sin^2 x = 1$
$\dfrac{d}{dx}(\cosh x) = \sinh x$	$\dfrac{d}{dx}(\cos x) = -\sin x$
$\dfrac{d}{dx}(\sinh x) = \cosh x$	$\dfrac{d}{dx}(\sin x) = \cos x$
$\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$	$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots$
$\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$	$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots$

These first two expressions for the circular functions were given in *Unit 5*, Subsection 4.2

The last two properties in the table can also be derived using Taylor's Theorem, given in Subsection 1.3. For example, if $f(x) = \cosh x$, the conditions of Taylor's Theorem are satisfied for any interval $[a, b]$ and we may write the $2n$th Taylor expansion about 0, with remainder, as

$$\cosh x = \cosh 0 + x \sinh 0 + \frac{x^2}{2!}\cosh 0 + \cdots + \frac{x^{2n}}{(2n)!}\cosh 0 + \frac{x^{2n+1}}{(2n+1)!}\sinh c_x \qquad (4)$$

where c_x lies between 0 and x.

Now $\cosh 0 = 1$ and $\sinh 0 = 0$. Hence Equation (4) simplifies to

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\sinh c_x.$$

$\cosh 0 = \dfrac{e^0 + e^0}{2} = 1,$

$\sinh 0 = \dfrac{e^0 - e^0}{2} = 0.$

As n becomes large we obtain the required expression for $\cosh x$, since for any x the remainder term approaches zero for large n. (We will not prove this, as it would take us too far from the main subject of the unit.)

In Exercise 3 you will be asked to obtain a formula for the length of a catenary. To do this you will need to use the following result.

Length of a curve

The length L of the curve $y = f(x)$ between x_0 and x_1 is given by

$$L = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

You can verify this result in the case of the straight line
$y = ax$
between $x_0 = 0$ and $x_1 = 1$ by showing that
$L = \sqrt{1 + a^2}$.

For most curves a numerical method would have to be used to evaluate L, but in the case of a catenary we can determine L exactly.

Exercise 1

Show that:

(i) $\cosh(-x) = \cosh x$

$\sinh(-x) = -\sinh x$

(ii) $\dfrac{d}{dx}(\cosh x) = \sinh x$

$\dfrac{d}{dx}(\sinh x) = \cosh x.$

Compare these results with the corresponding properties of sines and cosines.

Exercise 2

Use Taylor's Theorem to derive the Taylor series for $\sinh x$ as

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

Exercise 3

Determine the length of the catenary, between $x = -1$ and $x = 1$, given by

$$y = \frac{1}{a}\cosh ax.$$

[*Solutions to Exercises 1–3 on pp. 51–2*]

Now watch the television programme 'Catenaries numerical approximation'

TV 18

3.2 The rope problem

As I stated in the introduction to Section 3, the problem considered in the television programme is to find the displacement d at the middle of a length L of rope suspended between points two metres apart as in Figure 2.

In Subsection 3.1 I stated that the equation of a catenary is given by Equation (2) as

$$y = \frac{1}{a}\cosh ax. \tag{5}$$

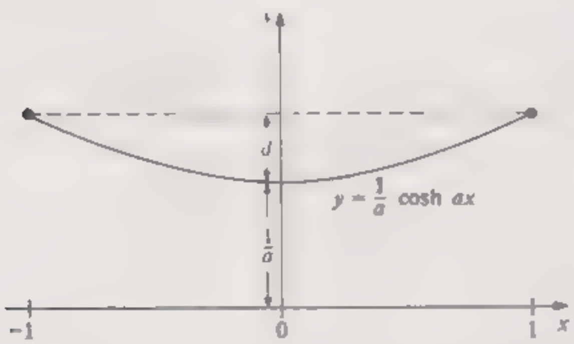


Figure 6. The catenary.

To obtain an expression for d we note that the height of the catenary above the x-axis at $x = 1$ is $d + \dfrac{1}{a}$ (see Figure 6), i.e.

$$y(1) = d + \frac{1}{a}.$$

Using the equation for the catenary (5) we have

$$y(1) = \frac{1}{a}\cosh a = d + \frac{1}{a}.$$

Hence

$$d = \frac{1}{a}(\cosh a - 1). \tag{6}$$

This gives d in terms of a, whereas we really require an expression for d in terms of L.

From Exercise 3 we know that the length of the catenary is given by

$$L = \frac{2}{a}\sinh a. \tag{7}$$

Equations (6) and (7) define d and L in terms of a, but unfortunately there is no way to establish a direct formula for d in terms of L. It is possible to use these formulae, with the help of the Newton-Raphson method, although it does involve a lot of work.

Example 1

Determine the displacement d if $L = 2.2\,\text{m}$.

Solution

From Equation (7) we have

$$2.2 = \frac{2}{a}\sinh a. \tag{8}$$

Rearranging this equation gives a as a root of the non-linear equation

$$f(a) = \sinh a - 1.1\,a = 0. \tag{9}$$

From the sketch graph of $f(a)$ in Figure 7 we can see that there are two non-negative roots for Equation (9). However, the root $a = 0$ is not a solution of Equation (8) and the value of a we require is approximately $a = 0.76$.

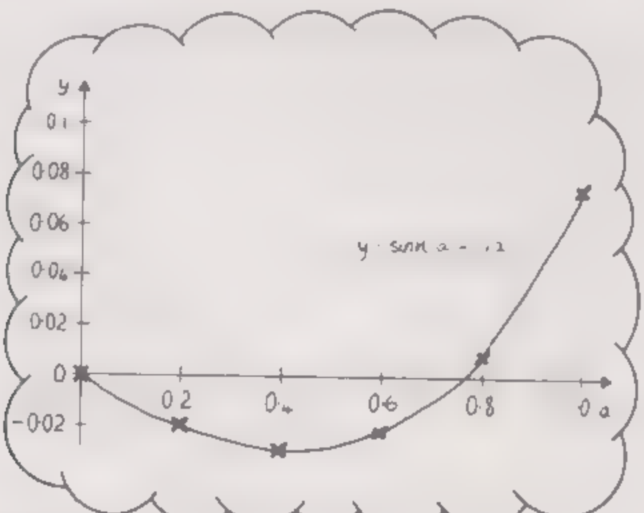


Figure 7. Sketch of $f(a) = \sinh a - 1.1a$.

The Newton-Raphson method, derived in Subsection 2.1, determines a root of the equation $f(x) = 0$ using the recurrence relation

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

with x_0 given.

For our function of a we have

$$f(a) = \sinh a - 1.1\,a \qquad \text{and} \qquad f'(a) = \cosh a - 1.1.$$

Hence the recurrence relation is

$$a_{r+1} = a_r - \left(\frac{\sinh a_r - 1.1\,a_r}{\cosh a_r - 1.1}\right).$$

With $a_0 = 0.76$ we have

$$\begin{aligned} a_1 &= 0.763\,424\,64 \\ a_2 &= 0.763\,400\,80 \\ &= a_3. \end{aligned}$$

After only 3 iterations the sequence has converged to the required root. With $a = 0.763\,400\,80$ we can now use Equation (6) to evaluate d as

$$d = \frac{1}{a}(\cosh a - 1) = 0.400\,601\,6.$$

Thus when $L = 2.2$ m, the displacement at the centre of the rope is given as $d = 0.401$ m correct to 3 decimal places.

In the above example we have managed to find d for a given length L. However, for each value of L we have to determine an initial value a_0, use the Newton-Raphson method, and then substitute the computed value of a into Equation (6) to determine the displacement.

In the second part of the television programme we look for a simple approximate relationship between d and L using Taylor polynomials, which works well for small displacements d. The advantage of a simple relationship is that it will be easier to make qualitative predictions about the behaviour of the rope for small displacements. Thus, although it is possible to obtain a more accurate approximate relationship between d and L, the simple one may be more useful in practice.

From the properties of hyperbolic functions we know that

$$\cosh a = 1 + \frac{a^2}{2!} + \frac{a^4}{4!} + \frac{a^6}{6!} + \cdots$$

Hence the 2nd Taylor polynomial about 0 for $\cosh a$ is

$$\cosh a \simeq 1 + \tfrac{1}{2}a^2.$$

Because the a^3 term is zero, this is actually the 3rd Taylor polynomial as well as the 2nd.

While a is reasonably close to zero this will provide a good approximation to $\cosh a$. In Equation (6) we have

$$\begin{aligned} d &= \frac{1}{a}(\cosh a - 1) \\ &\simeq \frac{1}{a}(1 + \tfrac{1}{2}a^2 - 1) \quad \text{(using the 2nd (or 3rd) Taylor polynomial about 0 for } \cosh a\text{)} \\ &= \tfrac{1}{2}a. \end{aligned} \tag{10}$$

Similarly we have

$$\sinh a = a + \frac{a^3}{3!} + \frac{a^5}{5!} + \cdots$$

and the 3rd Taylor polynomial about 0 for $\sinh a$ is

$$\sinh a \simeq a + \tfrac{1}{6}a^3.$$

Thus the equation for L is

$$\begin{aligned} L &= \frac{2}{a}\sinh a \\ &\simeq \frac{2}{a}(a + \tfrac{1}{6}a^3) \\ &= 2 + \tfrac{1}{3}a^2. \end{aligned} \tag{11}$$

Now Equation (11) can be rearranged to give a in terms of L as

$$a \simeq \sqrt{3(L-2)}.$$

Hence using Equation (10) the approximate displacement d^* is given by

$$d^* = \tfrac{1}{2}\sqrt{3(L-2)} = \sqrt{\tfrac{1}{4}(3L-6)}. \tag{12}$$

Now we have an explicit expression for d^* for any value of L. For example, if $L = 2.2$ we have

$$d^* = \sqrt{0.15} \simeq 0.387,$$

which is quite close to the more accurate value $d \sim 0.401$ m, obtained in Example 1

Comparing the approximate results, for different values of L, with the true results in Figure 8, we can see that for values of L slightly greater than 2 the approximation is very good, but as L increases the approximation gets worse

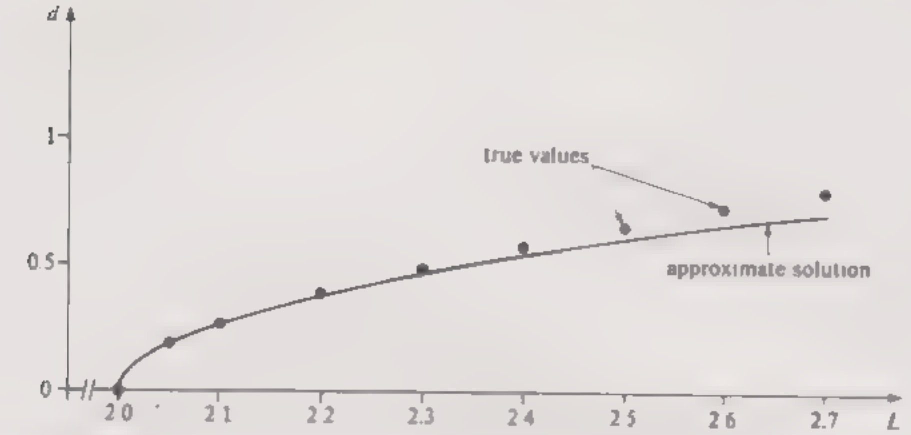


Figure 8 Graph comparing the approximate displacements with the true displacements.

Each of the true values was computed using Equations (6) and (7) and the Newton-Raphson method.

We conclude that our approximate method is a very useful way of determining the approximate displacement d^* when d is small Furthermore, the approximate relationship tells us something we might have otherwise missed For L slightly greater than 2 there is a very sharp increase in the displacement For example, if $L = 2.001$ we find that the displacement $d \simeq 0.027$ m i.e. if L is increased by 1 mm from the zero-displacement position then the displacement will be 2.7 cm, 27 times as big. Hence the problem of finding d given L is absolutely ill-conditioned for L close to 2 m The ill-conditioning is demonstrated at the end of the television programme, illustrating that approximate relationships can give us information that would otherwise be difficult to deduce.

Exercise 4

Use the Newton-Raphson method to determine the displacement d, to an accuracy of 6 significant figures, when $L = 2.08$ m.

Exercise 5

Express a in terms of L if the 5th Taylor polynomial about $a = 0$ is used to approximate $\sinh a$ in Equation (7).

Exercise 6

Use the approximation derived in Exercise 5 together with Equation (6) to find the approximate displacement if $L = 2.08$ m.

[*Solutions to Exercises 4–6 on pp. 52–3*]

Summary of Section 3

1. The equation modelling a freely hanging rope suspended between two points is

$$\begin{aligned} y &= \frac{1}{a}\left(\frac{e^{ax} + e^{-ax}}{2}\right) \\ &= \frac{1}{a}\cosh ax, \end{aligned}$$

where a is a parameter which depends on (among other things) the length of the rope, and cosh is the **hyperbolic cosine** function. The curve with this equation is called a **catenary**.

2. Hyperbolic functions, given by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

have many properties which are similar to those of the circular functions $\sin x$ and $\cos x$. A list of these properties is given on page 31.

3. The length of a curve between x_0 and x_1 is given by

$$L = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

In most cases a numerical method would be required to evaluate this integral.

4. The displacement d for a freely hanging rope of length L suspended between two points 2 m apart is given by

$$d = \frac{1}{a}(\cosh a - 1)$$

where

$$L = \frac{2}{a}\sinh a.$$

For a given value of L, a can be determined using the Newton-Raphson method and this value can be used to determine d. An approximate relationship between d and L, obtained using the 3rd Taylor polynomials about $a = 0$ for $\cosh a$ and $\sinh a$, is given by

$$d^* = \sqrt{\tfrac{1}{4}(3L - 6)}.$$

For $2 \leqslant L \leqslant 2.4$ this gives approximations within 7% of the true solution.

4 Interpolation and integration

4.1 Interpolation

In Section 1 we derived the Taylor polynomials of degree n by assuming that we knew or could determine n derivatives of a function at a particular point α. However, in many problems the values of the derivatives are not known, but the data is given at a number of points $x_0, x_1, x_2, x_3, \ldots, x_n$. Diagrammatically we could represent this as in Figure 1.

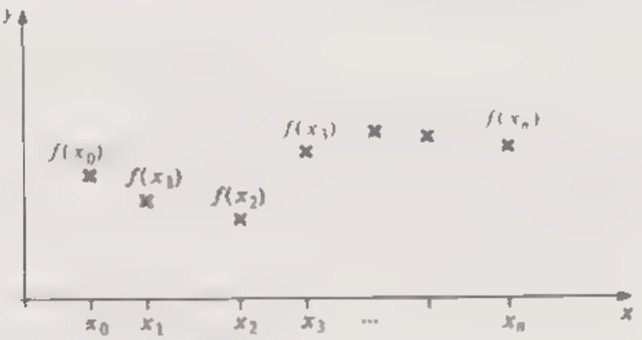


Figure 1. Data given at $n + 1$ points $x_0, x_1, \ldots, x_n$.

It is possible to show that there is just one polynomial of degree n which takes the desired value of f at each of the $n + 1$ points $x_0, x_1, \ldots, x_n$.

Once we have determined this polynomial we can use it to approximate the function at other values of x. This process of determining the approximate value of the function at a point x between x_0 and x_n is called **interpolation**. (The process of using the polynomial to determine approximate values outside this range is called *extrapolation*. Extrapolation is much less accurate than interpolation and we will not consider this topic here.) One method of computing the coefficients of the **interpolating polynomial of degree n**, $p(x)$, given as

$$p(x) = a_0 + a_1x + \ldots + a_nx^n,$$

is to set up a system of equations. Since $p(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$, we can write

$$\begin{aligned} p(x_0) &= a_0 + a_1x_0 + \ldots + a_nx_0^n = f(x_0) \\ p(x_1) &= a_0 + a_1x_1 + \ldots + a_nx_1^n = f(x_1) \\ &\vdots \\ p(x_n) &= a_0 + a_1x_n + \ldots + a_nx_n^n = f(x_n) \end{aligned}$$

This is just a system of linear equations, like those you met in *Unit 9, Simultaneous linear algebraic equations*, for the $n + 1$ unknowns $a_0, a_1, \ldots, a_n$. In matrix form the problem is to find $a_0, a_1, \ldots, a_n$ which satisfy the equations

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n \\ 1 & x_1 & x_1^2 & \ldots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \ldots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

Note that in this problem we know the values of $x_0, x_1, \ldots, x_n$ and $f(x_0), f(x_1), \ldots, f(x_n)$ and we wish to determine $a_0, a_1, \ldots, a_n$.

Example 1

Determine the interpolating quadratic polynomial which passes through the three points (0, 1), (1, 2) and (2, 4).

Solution

Let the interpolating polynomial be given by

$$p(x) = a_0 + a_1x + a_2x^2.$$

The three equations corresponding to the three data points are

$$\begin{aligned} p(0) &= a_0 + a_1(0) + a_2(0^2) = 1 \\ p(1) &= a_0 + a_1(1) + a_2(1^2) = 2 \\ p(2) &= a_0 + a_1(2) + a_2(2^2) = 4 \end{aligned}$$

giving

$$\begin{aligned} a_0 \qquad\qquad &= 1 \\ a_0 + a_1 + a_2 &= 2 \\ a_0 + 2a_1 + 4a_2 &= 4 \end{aligned}$$

i.e.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Using Gaussian elimination, as described in *Unit 9*, we reduce this matrix to its upper triangular form as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

whence

$$a_2 = \tfrac{1}{2},\ a_1 = \tfrac{1}{2} \text{ and } a_0 = 1$$

Thus the interpolating polynomial is

$$p(x) = 1 + \tfrac{1}{2}x + \tfrac{1}{2}x^2.$$

Given a large set of points it is not always desirable to compute the interpolating polynomial of corresponding degree. This process can be very time-consuming, prone to the build-up of rounding errors in computation, and the resulting polynomial tedious to evaluate using a calculator. More frequently, given a value of x at which we need to approximate the function, we construct a polynomial using just a few of the data points.

Example 2

Here is a set of data obtained from four-figure tables for $\log_e x$.

x_n	$\log_e x_n$
1.11	0.1044
1.12	0.1133
1.13	0.1222
1.14	0.1310
1.15	0.1398
1.16	0.1484

Use an interpolating polynomial of degree 2 to obtain an approximation to $\log_e (1.134)$.

Solution

For a quadratic polynomial we need to specify 3 points x_0, x_1 and x_2. We choose the three points closest to $x = 1.134$, namely

$$x_0 = 1.12, \qquad x_1 = 1.13, \qquad x_2 = 1.14.$$

If $p(x) = a_0 + a_1 x + a_2 x^2$, we obtain the following equations for a_0, a_1 and a_2

$$p(1.12) = a_0 + 1.12a_1 + (1.12)^2\, a_2 = 0.1133$$
$$p(1.13) = a_0 + 1.13a_1 + (1.13)^2\, a_2 = 0.1222$$
$$p(1.14) = a_0 + 1.14a_1 + (1.14)^2\, a_2 = 0.1310$$

i.e.

$$\begin{bmatrix} 1 & 1.12 & 1.2544 \\ 1 & 1.13 & 1.2769 \\ 1 & 1.14 & 1.2996 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.1133 \\ 0.1222 \\ 0.1310 \end{bmatrix}.$$

Using the computer package SIMLIN, I obtained the solution

$$a_0 = -1.516\,35, \qquad a_1 = 2.015\,08, \qquad a_2 = -0.500\,04.$$

Hence

$$p(x) = -1.516\,35 + 2.015\,08\, x - 0.500\,04\, x^2,$$

giving

$$p(1.134) = 0.1257 \text{ to 4 significant figures.}$$

It is worth pointing out that the problem of evaluating the coefficients of p in the above example is highly ill-conditioned. To illustrate this I repeated the calculations with the right-hand side values correct to five significant figures. The new set of equations to solve is

$$\begin{bmatrix} 1 & 1.12 & 1.2544 \\ 1 & 1.13 & 1.2769 \\ 1 & 1.14 & 1.2996 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.113\,33 \\ 0.122\,22 \\ 0.131\,03 \end{bmatrix}.$$

SIMLIN this time gives

$$a_0 = -1.388\,63, \qquad a_1 = 1.789\,08 \qquad \text{and} \qquad a_2 = -0.400\,04,$$

so that

$$p(x) = -1.388\,63 + 1.789\,08\,x - 0.400\,04\,x^2.$$

This is completely different from the polynomial obtained in Example 2, and the difference is caused by very small changes in the data. The reason for this ill-conditioning is partly due to the fact that the rows of the matrix are almost linearly dependent (you can verify this if you wish!). However, if we use the new polynomial to evaluate an approximation to $\log_e(1.134)$ we get

$$p(1.134) = 0.125\,75 \text{ to 5 significant figures,}$$

and this answer is almost identical to the answer obtained in Example 2. The true solution (to 8 decimal places) is

$$\log_e(1.134) = 0.125\,751\,21,$$

indicating that the answers we obtained are remarkably good. Although the polynomials are completely different, the difference in their values for $1.12 \leqslant x \leqslant 1.14$ is less than 0.000 04, i.e. approximately the same as the differences in the data!

What has happened is that although the problem of computing the coefficients of the interpolating polynomial is ill-conditioned, the problem of determining the approximate value for $\log_e(1.134)$ is well-conditioned. This phenomenon occurs very frequently in interpolation problems where the x-values are close together. You need to be aware that this can occur, although it almost always makes no difference to the final answer.

Exercise 1

Determine the quadratic whose graph passes through the three points $(-1,2)$, $(0,3)$ and $(1,0)$.

Exercise 2

Given the data in Example 2, use an interpolating polynomial of degree 1 to compute an approximation to $\log_e(1.134)$.

[*Solutions to Exercises 1 and 2 on p. 53*]

In Section 1, when we discussed Taylor polynomials, we stated Taylor's Theorem which could be used to determine the error in the approximation. In the same way there is a very useful theorem which can be used to determine the error in the interpolating polynomial. In this theorem we assume that all arithmetic has been carried out exactly and there are no rounding errors in the values $f(x_0), f(x_1), \ldots, f(x_n)$.

Theorem 4.1
You are given a function $f(x)$ and an interpolating polynomial $p(x)$ of degree $\leqslant n$ such that

$$p(x_i) = f(x_i) \qquad \text{for } i = 0, 1, \ldots, n,$$

where the numbers $x_0, x_1, \ldots, x_n$ are arranged in increasing order. Then the error, $\varepsilon(x) = p(x) - f(x)$, where x lies between x_0 and x_n, is given by

$$\varepsilon(x) = -(x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(c_x)}{(n+1)!} \qquad (1)$$

where c_x is some number satisfying $x_0 \leqslant c_x \leqslant x_n$.

You can see that the presence of the product $(x - x_0)(x - x_1)\ldots(x - x_n)$ ensures that

$$\varepsilon(x_i) = 0 \qquad \text{for } i = 0, 1, \ldots, n$$

as required, while the term

$$\frac{f^{(n+1)}(c_x)}{(n+1)!}$$

is very similar to the corresponding term in Taylor's Theorem.

The following procedure uses Theorem 1 to compute an error bound at a particular value of x between x_0 and x_n.

Error bounds for interpolating polynomials

1. Let $p(x)$ be the polynomial of degree $\leqslant n$ which interpolates the function $f(x)$ at $x_0, x_1, \ldots, x_n$, where $x_0 < x_1 < \ldots < x_n$.
2. The error $\varepsilon(x) = p(x) - f(x)$ is given by
$$\varepsilon(x) = -(x - x_0)(x - x_1)\ldots(x - x_n)\frac{f^{(n+1)}(c_x)}{(n+1)!}$$
where c_x lies between x_0 and x_n.
3. Determine a value for M such that
$$|f^{(n+1)}(c)| \leqslant M$$
for all values of c between x_0 and x_n.
4. A bound on the error $\varepsilon(x)$ is then
$$|\varepsilon(x)| \leqslant \frac{|(x - x_0)(x - x_1)\ldots(x - x_n)|\, M}{(n+1)!}$$

To see how this procedure can be used, we calculate an error bound in the following example.

Example 3

(i) Show that $|1/c^2| \leqslant 1$ for all values of c between 1 and 2.

(ii) The linear approximation to $\log_e x$ whose graph passes through the points (1, 0) and (2, 0.6931) is given by
$$p(x) = 0.6931x - 0.6931.$$
Determine an error bound at $x = 1.5$ and compare this bound with the actual error at $x = 1.5$.

The only error in the calculation of $p(x)$ is in giving $\log_e 2$ to only 4 decimal places.

Solution

(i) Since $c \geqslant 1$, we have $0 \leqslant 1/c \leqslant 1$, and so $|1/c^2| \leqslant 1$.

(ii) Using the procedure for computing error bounds, we note that with $x_0 = 1$ and $x_1 = 2$ the error function is given by
$$\varepsilon(x) = -(x - 1)(x - 2)\frac{f''(c_x)}{2}$$
where c_x lies between 1 and 2.

Now $f(x) = \log_e x$, and so
$$f'(x) = 1/x \quad \text{and} \quad f''(x) = -1/x^2.$$
Hence from part (i) an upper bound for $|f''(c)| = |1/c^2|$ where $1 \leqslant c \leqslant 2$ is given by
$$M = 1.$$
Thus $|\varepsilon(x)| \leqslant \dfrac{|(x-1)(x-2)|}{2}$, and at $x = 1.5$ we have
$$|\varepsilon(1.5)| \leqslant \frac{|(1.5 - 1)(1.5 - 2)|}{2} = 0.125.$$
Thus an error bound for this approximation is given by 0.125. The actual error is given by
$$\begin{aligned} p(1.5) - \log_e 1.5 &= 0.346\,55 - 0.405\,47 \\ &= -0.058\,92 \end{aligned}$$
so that $|\varepsilon(1.5)| = 0.058\,92$. This is certainly less than the error bound 0.125.

Exercise 3

(i) Show that, for $-1 \leqslant c \leqslant 1$, we have $|e^c| \leqslant e = 2.7183\ldots$.

(ii) The function $f(x) = e^x$ is to be approximated by the interpolating quadratic polynomial which interpolates f at $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Determine a bound for the error in this quadratic at $x = 0.5$, ignoring rounding errors. (Note that you are *not* asked to compute the interpolating quadratic polynomial.)

[*Solution on p. 53*]

4.2 Methods of numerical integration

In the last subsection we looked at methods of interpolation. In this final subsection we examine how these interpolation methods can be used to derive approximations to the integral

$$I = \int_a^b f(x)\,dx$$

where f is some given function.

By integrating the interpolating polynomial, which is a fairly straightforward process, we can obtain formulae for the approximation of the integral. Such formulae are known as **Newton-Cotes formulae**. The methods derived here, **Euler's integration method**, the **trapezoidal method** and **Simpson's method**, will be used in the next unit to solve differential equations, but they are clearly useful in their own right as well.

Euler's method

The reason for calling it Euler's method is that it is closely related to Euler's method for differential equations. See *Unit 19*.

Suppose we want to approximate the integral

$$\int_{x_0}^{x_1} f(x)dx.$$

Figure 2 shows one simple approximation: we replace the function $f(x)$ by the constant $f(x_0)$ for all values of x between x_0 and x_1. This is equivalent to using a very crude 'approximating polynomial', given by

$$p(x) = f(x_0).$$

The integral

$$I = \int_{x_0}^{x_1} f(x)dx$$

is the area under the curve $y = f(x)$ between x_0 and x_1. We approximate this area by the area under the approximating polynomial as

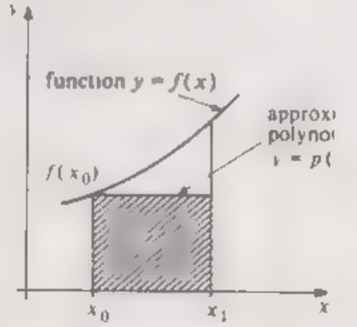


Figure 2. Euler's method.

$$I \simeq \int_{x_0}^{x_1} p(x)dx = \int_{x_0}^{x_1} f(x_0)dx = (x_1 - x_0)f(x_0)$$
$$= hf(x_0) \qquad \text{where } h = x_1 - x_0,$$

i.e. the area under the curve $y = f(x)$ is approximated by the area of the rectangle of height $f(x_0)$ and width h.

Clearly this is a very crude approximation, and Euler's method would rarely be used in this form. However, we can get more accurate results by using a **composite form** of the method. Suppose we wish to approximate the integral of f over the interval $[a, b]$. Then we divide this interval into n subintervals of equal width h and use Euler's method on each subinterval (see Figure 3).

The approximation to the integral is then

$$\begin{aligned}\int_a^b f(x)dx &= \int_{x_0}^{x_1} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \\ &\simeq hf(x_0) + hf(x_1) + \cdots + hf(x_{n-1}) \\ &= h\big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big). \qquad (2)\end{aligned}$$

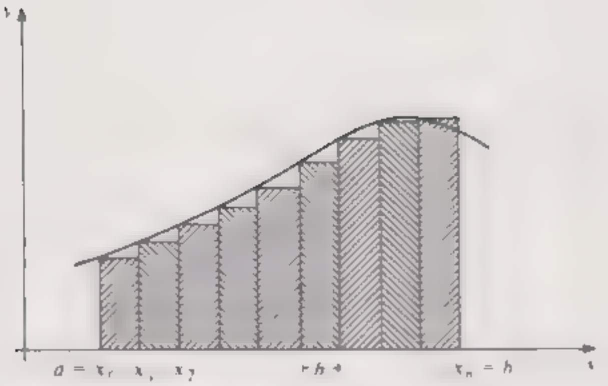


Figure 3. Euler's method in composite form.

Example 4

Determine the approximate value of

$$I = \int_1^2 \log_e x\,dx$$

using the composite Euler's method with $h = 0.1$.

Solution

Using Equation (2) with $h = 0.1$, we have

$$I \simeq 0.1\ (\log_e 1 + \log_e 1.1 + \log_e 1.2 + \cdots + \log_e 1.9)$$
$$= 0.351\,22.$$

For this example the true solution is given by

$$\int_1^2 \log_e x\,dx = [x \log_e x - x]_1^2$$
$$= 2\log_e 2 - 2 + 1$$
$$= 0.386\,29.$$

The approximation is not particularly good, but from Figure 3 we can deduce that if we use a very small interval width h we can achieve much better accuracy.

Euler's method in composite form

1. You are asked to obtain a value of the integral

$$I = \int_a^b f(x)dx.$$

2. Choose values for h and n such that

$$b = a + nh.$$

3. Approximate I using Euler's method in composite form as

$$I \simeq h\big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big)$$

where $x_r = a + rh$.

Exercise 4

Use Euler's method in composite form with $h = 0.2$ to approximate the integral of e^x between $x = 0$ and $x = 1$.

[*Solution on p. 53*]

The trapezoidal method

Consider the linear interpolating polynomial approximation to the function $y = f(x)$ which passes through the data points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

The integral

$$I = \int_{x_0}^{x_1} f(x)dx$$

is approximated by the area under the graph of the interpolating polynomial. For linear interpolation this area is a **trapezium** - a quadrilateral with two opposite sides parallel. The area of this trapezium is the product of the width $h = x_1 - x_0$ times the average height, which is $\frac{1}{2}(f(x_0) + f(x_1))$. The approximation to the integral I is thus

$$I \simeq \tfrac{1}{2}h\big(f(x_0) + f(x_1)\big). \tag{3}$$

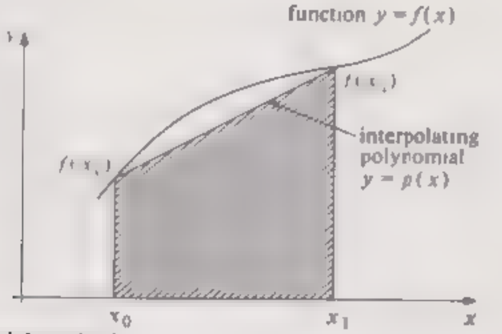


Figure 4. The trapezoidal method.

This method of approximating the integral is known as the **trapezoidal method**. We would expect the trapezoidal method to give more accurate results than Euler's method, since the trapezium fits the area under the curve better than the rectangle does.

Example 5

Determine an approximate value of

$$I = \int_1^2 \log_e x\,dx$$

using the trapezoidal method.

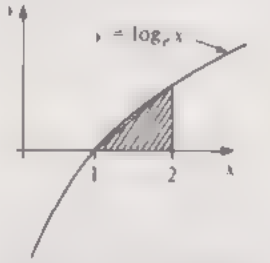


Figure 5. The trapezoidal method for Example 5.

Solution

Equation (3) gives the approximation as

$$I \simeq \tfrac{1}{2}(\log_e 1 + \log_e 2) = 0.346\,57.$$

The correct solution is given in Example 4 as $I = 0.386\,29$. Even with this fairly crude method we have obtained a reasonable approximation to the integral.

The trapezoidal method is most commonly used in **composite form**. That is, if we wish to approximate the integrand f over the interval $[a, b]$, then we divide this interval into n subintervals of equal width and use the trapezoidal method on each subinterval, as shown in Figure 6.

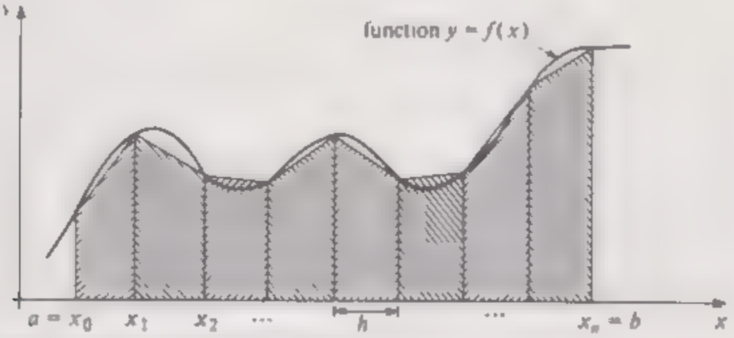


Figure 6. The composite trapezoidal method.

$$\begin{aligned} I = \int_a^b f(x)dx &\simeq \tfrac{1}{2}h\big(f(x_0) + f(x_1)\big) + \tfrac{1}{2}h\big(f(x_1) + f(x_2)\big) + \\ &\quad + \tfrac{1}{2}h\big(f(x_{n-1}) + f(x_n)\big) \\ &= h\big(\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\big). \end{aligned} \tag{4}$$

Example 6

Determine the approximate value of

$$I = \int_1^2 \log_e x \, dx$$

using the composite trapezoidal method with $h = 0.25$.

Solution

Equation (4) with $f(x) = \log_e x$ and $h = 0.25$ gives

$$I \simeq 0.25\left(\tfrac{1}{2}\log_e 1 + \log_e 1.25 + \log_e 1.5 + \log_e 1.75 + \tfrac{1}{2}\log_e 2\right)$$
$$= 0.383\,70.$$

This is almost correct to 2 decimal places, since the true solution is $I = 0.386\,29$.

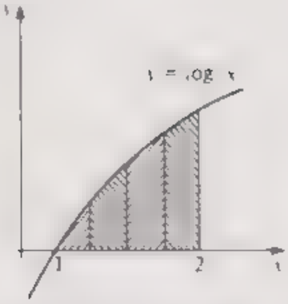


Figure 7. The composite trapezoidal method applied in Example 6.

The trapezoidal method in composite form

1. You are asked to find a value of

$$I = \int_a^b f(x)dx.$$

2. Choose values for h and n such that

$$b = a + nh.$$

3. Approximate I using the trapezoidal method in composite form as

$$I \simeq h\left(\tfrac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)\right) \qquad (5)$$

where $x_r = a + rh$.

Exercise 5

Use the composite trapezoidal method (Equation (5)) with $h = 0.2$ to estimate the integral of e^x between $x = 0$ and $x = 1$. Compare your answer with the true solution and the solution obtained in Exercise 4.

[*Solution on p. 53*]

Simpson's method

Our third method of numerical integration is known as **Simpson's method**. It is derived by interpolating three equally spaced data points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ by a quadratic polynomial and then integrating this quadratic, as in Figure 8. Since it can allow for the curvature of the graph of the function f, we could expect Simpson's method to give much more accurate results than Euler's method or the trapezoidal method.

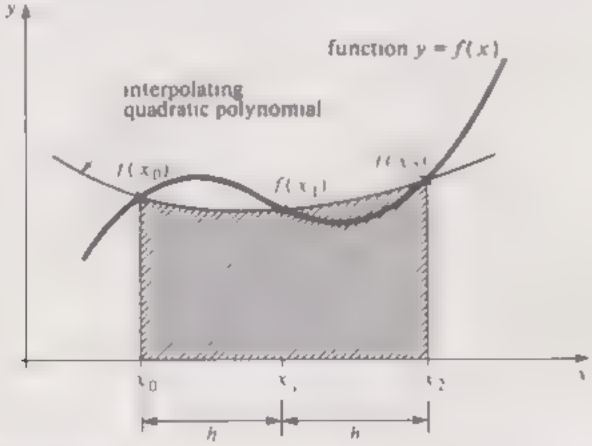


Figure 8. Simpson's method.

To simplify the derivation of the formula for Simpson's method we make a change of variable, writing

$$s = x - x_1, \qquad (6)$$

and we denote the quadratic interpolating polynomial, written in terms of the variable s, by

Do not spend a lot of time working through the derivation of Equation (10).

$$q(s) = a + bs + cs^2.$$

From Equation (6) and the conditions determining the interpolating polynomial we see that:

when $x = x_0$ we have $s = -h$ and $q(-h) = f(x_0)$,
when $x = x_1$ we have $s = 0$ and $q(0) = f(x_1)$,
when $x = x_2$ we have $s = h$ and $q(h) = f(x_2)$.

Hence the conditions determining the coefficients a, b, c in the polynomial $q(s)$ are

$$q(-h) = a - bh + ch^2 = f(x_0) \tag{7}$$
$$q(0) = a = f(x_1) \tag{8}$$
$$q(h) = a + bh + ch^2 = f(x_2). \tag{9}$$

Equation (8) gives $a = f(x_1)$. Thus Equations (7) and (9) can be rewritten as

$$-bh + ch^2 = f(x_0) - f(x_1)$$
$$bh + ch^2 = f(x_2) - f(x_1).$$

These two equations can now be solved for b and c, giving

$$b = \frac{f(x_2) - f(x_0)}{2h}$$

and

$$c = \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}.$$

Thus

$$q(s) = f(x_1) + \left(\frac{f(x_2) - f(x_0)}{2h}\right)s + \left(\frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}\right)s^2.$$

To approximate the integral of $f(x)$ between x_0 and x_2 we write

$$\begin{aligned}
\int_{x_0}^{x_2} f(x)dx &\simeq \int_{x_0}^{x_2} p(x)dx \\
&= \int_{-h}^{h} q(s)ds \\
&= \int_{-h}^{h} \left\{ f(x_1) + \left(\frac{f(x_2) - f(x_0)}{2h}\right)s \right. \\
&\qquad \left. + \left(\frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}\right)s^2 \right\} ds \\
&= \left[f(x_1)s + \left(\frac{f(x_2) - f(x_0)}{2h}\right)\frac{s^2}{2} \right. \\
&\qquad \left. + \left(\frac{f(x_2) - 2f(x_1) + f(x_0)}{2h^2}\right)\frac{s^3}{3} \right]_{-h}^{h} \\
&= 2hf(x_1) + 0 + \frac{h}{3}(f(x_2) - 2f(x_1) + f(x_0)).
\end{aligned}$$

When $x = x_0$, $s = -h$, and when $x = x_2$, $s = h$. Equation (6) gives $ds = dx$.

Hence

$$\int_{x_0}^{x_2} f(x)dx \simeq \frac{h}{3}(f(x_2) + 4f(x_1) + f(x_0)). \tag{10}$$

Now that we have the formula for Simpson's method, given by Equation (10), we can simply use this to approximate integrals.

Example 7

Determine an approximation to the integral

$$I = \int_1^2 \log_e x \, dx$$

using Simpson's method.

Solution

To use Simpson's method we need two equal intervals of width $h = \frac{1}{2}$. From Equation (10) we have, since $h = \frac{1}{2}(x_2 - x_0) = \frac{1}{2}$,

$$\begin{aligned} I &\simeq \tfrac{1}{6}(\log_e 1 + 4\log_e 1.5 + \log_e 2) \\ &= 0.385\,83 \end{aligned}$$

and this is correct to three decimal places.

We have not drawn a figure in this case, as there would be no visible difference between the graph of $y = \log_e x$ and the interpolating quadratic.

Like Euler's method and the trapezoidal method, Simpson's method can be used **in composite form**. We divide the interval $[a, b]$ into an even number of subintervals and approximate the integral as the area under the quadratic for each pair of subintervals.

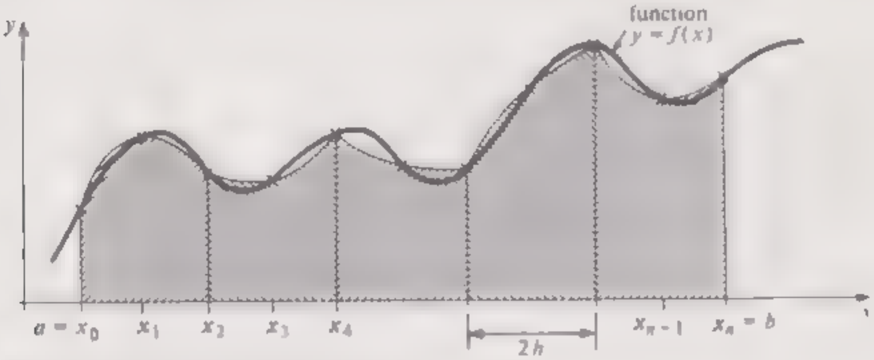


Figure 9. The composite Simpson's method.

$$\begin{aligned} \int_a^b f(x)dx &\simeq \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) + \frac{h}{3}(f(x_2) + 4f(x_3) + f(x_4)) + \cdots \\ &\quad + \frac{h}{3}(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) \\ &= \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \\ &\quad + 4f(x_{n-1}) + f(x_n)). \end{aligned} \qquad (11)$$

The composite Simpson's method is one of the most popular methods for numerical integration. We summarize the procedure in the following box.

Simpson's method in composite form

1. You are asked to obtain a value for the integral

$$I = \int_a^b f(x)dx.$$

2. Choose values for h and n such that n is even and

$$b = a + nh.$$

3. Approximate I using Simpson's method in composite form as

$$\begin{aligned} I &\simeq \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\ &\quad + 4f(x_{n-1}) + f(x_n)) \end{aligned}$$

where $x_r = a + rh$.

Example 8

Determine an approximate value for the integral

$$I = \int_1^2 \log_e x\,dx$$

using the composite Simpson's method with $h = \frac{1}{4}$.

Solution

With $h = \frac{1}{4}$, Equation (11) gives

$$\begin{aligned} I &\simeq \tfrac{1}{12}(\log_e 1 + 4\log_e 1.25 + 2\log_e 1.5 + 4\log_e 1.75 + \log_e 2) \\ &= 0.386\,260 \end{aligned}$$

and this solution is correct to four decimal places.

Exercise 6

(i) Use Simpson's method to obtain an approximation to the integral of e^x between $x = 0$ and $x = 1$.

(ii) Use the composite Simpson's method for the same integral with $h = \frac{1}{4}$ and $h = \frac{1}{8}$. Check the accuracy of your solutions.

Exercise 7

The length of the curve $y = \sin x$ between 0 and $\frac{\pi}{2}$ is given by

$$L = \int_0^{\pi/2} (1 + \cos^2 x)^{1/2}\,dx.$$

See Subsection 3.1.

Use the composite Simpson's method with $h = \frac{\pi}{8}$ to find the approximate length

[*Solutions to Exercises 6 and 7 on p. 53*]

Summary of Section 4

1. Given a set of data $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$, **interpolation** refers to any method of estimating the value of $f(x)$ at an intermediate point x by constructing an interpolating function whose graph passes through these points and then evaluating this function at x.

2. To construct an **interpolating polynomial** p of degree $\leqslant n$ given by

   $$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

   which passes through the points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x^n))$, we solve the $n + 1$ linear equations for $a_0, \ldots, a_n$ given by

   $$p(x_i) = f(x_i), \quad i = 0, 1, \ldots, n.$$

3. The **interpolation error** for a polynomial of degree n is given in Theorem 4.1 A bound for this error can be found using the procedure on page 40.

4. **Euler's method** for approximating the integral of f between x_0 and x_1 is given by

   $$\int_{x_0}^{x_1} f(x)dx \simeq hf(x_0).$$

   Euler's method in composite form for approximating the integral of f between x_0 and x_n is given by

   $$\int_{x_0}^{x_n} f(x)dx \simeq h\left(f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right)$$

   where $h = x_{i+1} - x_i, \quad i = 0, \ldots, n-1.$

5. The **trapezoidal method** for approximating the integral of f between x_0 and x_1 is given by

$$\int_{x_0}^{x_1} f(x)dx \simeq \frac{h}{2}(f(x_0)+f(x_1))$$

where $h = x_1 - x_0$.

The **composite trapezoidal method** for approximating the integral of f between x_0 and x_n is given by

$$\int_{x_0}^{x_n} f(x)dx \simeq h\left(\tfrac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\tfrac{1}{2}f(x_n)\right)$$

where $h = x_{i+1} - x_i$, $\quad i = 0, \ldots, n-1$.

6. **Simpson's method** for approximating the integral of f between x_0 and x_2 is given by

$$\int_{x_0}^{x_2} f(x)dx \simeq \frac{h}{3}(f(x_0)+4f(x_1)+f(x_2))$$

where $h = x_2 - x_1 = x_1 - x_0$.

The **composite Simpson's method** for approximating the integral of f between x_0 and x_n (n even) is given by

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{h}{3}(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2}) + 4f(x_{n-1})+f(x_n))$$

where $h = x_{i+1} - x_i$, $\quad i = 0, \ldots, n-1$.

Simpson's method is usually much more accurate than the trapezoidal method or Euler's method.

5 End of unit exercises

Exercise 1

(i) Show that the maximum value of $2\tan c\sec^2 c$ is 0.20 to 2 decimal places for $0 \leqslant c \leqslant 0.1$.

(ii) Determine the linear Taylor polynomial about 0 for $\tan x$ and give an error bound on the approximation of $\tan x$ by this polynomial for $0 \leqslant x \leqslant 0.1$.

Hint:
$\sec^2 x = 1 + \tan^2 x$
$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\sec x = \sec x \tan x$.

Exercise 2

Determine the two positive roots of the equation

$$f(x) = e^x - 3x = 0$$

to 5 decimal places.

Exercise 3

Use the Taylor series method of order 2 with $h = 0.1$ to obtain approximations to the solution of the differential equation

$$y' = xy + 1 \qquad \text{with } y(0) = 1$$

at $x = 0.1$ and $x = 0.2$. Give your approximations to 3 decimal places.

Exercise 4

(i) You are given just the following information about a function f:

$$f(1) = 1 \text{ and } f(2\tfrac{1}{4}) = 1\tfrac{1}{2}.$$

Use an interpolating polynomial of degree one to approximate the function at $x = 1.44$.

(ii) Verify that for $1 \leqslant c \leqslant 2\frac{1}{4}$ we have $|\frac{1}{4}c^{-3/2}| \leqslant \frac{1}{4}$.

(iii) The function f in this problem is actually given by

$$f(x) = \sqrt{x}.$$

Determine an error bound for the approximation at $x = 1.44$ and compare this bound with the actual error.

Exercise 5

Use Simpson's method in composite form with $h = 0.25$ to approximate the integral of $\tan x$ between 0 and 1.

Exercise 6

Use Simpson's method in composite form with $h = 0.25$ to determine the approximate length of the curve $y = e^x$ between $x = 0$ and $x = 1$.

[Solutions to Exercises 1–6 on pp. 53–5]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1. (i) Differentiating term by term, we have

$$p'(x) = 8x^3 + 9x^2 + 2x - 7.$$

(ii) Integrating term by term gives

$$\int p(x)dx = \frac{2}{5}x^5 + \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{7}{2}x^2 + 2x + C$$

where C is the constant of integration.

(iii) Using nested multiplication, we have

$$\begin{aligned} u_0 &= 2 \\ u_1 &= 0.29u_0 + 3 = 3.58 \\ u_2 &= 0.29u_1 + 1 = 2.0382 \\ u_3 &= 0.29u_2 - 7 = -6.408\,922 \\ u_4 &= 0.29u_3 + 2 = 0.141\,412\,62. \end{aligned}$$

Hence $p(0.29) = 0.141\,412\,62$.

(iv) From (i), $p'(x) = 8x^3 + 9x^2 + 2x - 7$.
Using nested multiplication for this cubic polynomial gives

$$\begin{aligned} u_0 &= 8 \\ u_1 &= 0.77u_0 + 9 = 15.16 \\ u_2 &= 0.77u_1 + 2 = 13.6732 \\ u_3 &= 0.77u_2 - 7 = 3.528\,364. \end{aligned}$$

Hence $p'(0.77) = 3.528\,364$.

2. Since $f(x) = (1+x)^{1/2}$, we have

$$f'(x) = \tfrac{1}{2}(1+x)^{-1/2}$$
$$f''(x) = (\tfrac{1}{2})(-\tfrac{1}{2})(1+x)^{-3/2} = -\tfrac{1}{4}(1+x)^{-3/2}.$$

When $x = 0$, $(1+x)^k = 1$ for any k. Thus

$$f(0) = 1, f'(0) = \tfrac{1}{2} \text{ and } f''(0) = -\tfrac{1}{4}.$$

Hence the 2nd Taylor polynomial is

$$p(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2.$$

At $x = 0.44$ we evaluate $p(x)$ using nested multiplication as

$$\begin{aligned} u_0 &= -0.125 \\ u_1 &= 0.44u_0 + 0.5 = 0.445 \\ u_2 &= 0.44u_1 + 1 = 1.1958, \end{aligned}$$

i.e. $p(0.44) = 1.1958$

This is correct to two decimal places, since $(1 + 0.44)^{1/2} = 1.2$.

3. $f(x) = \log_e(1+x)$. Differentiating gives

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = -\frac{1}{(1+x)^2},$$
$$f'''(x) = \frac{2}{(1+x)^3}, \ldots, \quad f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}.$$

At $x = 0$ we have

$$f(0) = \log_e 1 = 0, \quad f'(0) = 1, \quad f''(0) = -1$$
$$f'''(0) = 2, \ldots, \quad f^{(n)}(0) = (-1)^{n+1}(n-1)!$$

Hence the nth Taylor polynomial about 0 for $\log_e x$ is

$$\begin{aligned} p(x) &= x + \frac{(-1)x^2}{2!} + \frac{2x^3}{3!} + \cdots + \frac{(-1)^{n+1}(n-1)!\,x^n}{n!} \\ &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots + \frac{(-1)^{n+1}x^n}{n} \end{aligned}$$

as required.

4. (i) Since $0 \leqslant c \leqslant 0.5$, we have $1 \leqslant 1 + c \leqslant 1.5$, and hence $|1 + c| \leqslant 1.5$

(ii) $f(x) = (1+x)^5$. Thus

$$f'(x) = 5(1+x)^4, \quad f''(x) = 20(1+x)^3,$$
$$f'''(x) = 60(1+x)^2 \quad \text{and} \quad f^{(4)}(x) = 120(1+x).$$

The error function for the 3rd Taylor polynomial is given by

$$\varepsilon(x) = -\frac{x^4}{4!}f^{(4)}(c_x) = -5x^4(1+c_x)$$

where c_x lies between 0 and x.

Now we are given that $0 \leqslant x \leqslant 0.5$, so it follows that $0 \leqslant c_x \leqslant 0.5$ and hence (by part (i)) that $|1 + c_x| \leqslant 1.5$. Using this result to calculate a bound for $\varepsilon(x)$, we obtain

$$|\varepsilon(x)| \leqslant 5x^4 \times 1.5 = 7.5x^4 \quad (0 \leqslant x \leqslant 0.5)$$

which is the required result.

5. (i) We know that $-1 \leqslant \sin c \leqslant 1$ for all c, and hence $|\sin c| \leqslant 1$.

(ii) Let $f(x) = \sin x$. Differentiating gives

$$f'(x) = \cos x, \quad f''(x) = -\sin x,$$
$$f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x,$$
$$f^{(5)}(x) = \cos x \quad \text{and} \quad f^{(6)}(x) = -\sin x.$$

At $x = 0$, $\sin x = 0$ and $\cos x = 1$. Hence

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0,$$
$$f'''(0) = -1, \quad f^{(4)}(0) = 0 \quad \text{and} \quad f^{(5)}(0) = 1.$$

Hence the 5th Taylor polynomial about 0 for $\sin x$ is

$$p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

The error is given using Taylor's Theorem, as

$$\varepsilon(x) = \frac{x^6}{6!}\sin c_x \qquad \text{where } c_x \text{ lies between 0 and } x.$$

From part (i) we know that $|\sin c_x| \leqslant 1$. It follows that

$$|\varepsilon(x)| \leqslant \frac{x^6}{6!}$$

Thus, at $x = 1$, we have

$$|\varepsilon(1)| \leqslant \frac{1}{6!} = 0.001\,388\,89.$$

Our error bound for the 5th Taylor polynomial about 0 approximating $\sin x$ at $x = 1$ is thus 0.001 39, to three significant figures. (There is no point in quoting errors and error bounds to a large number of significant figures.)

The actual value of sin 1 is 0.841 470 99, and of the Taylor polynomial is 0.841 666 67, so the actual error is only 0.000 196 (again to three figures).

Solutions to the exercises in Section 2

1. From the graph the three roots are approximately -0.5, -3.5 and -9.5. Using these values as initial estimates for the Newton-Raphson method with the recurrence relation

$$x_{r+1} = x_r - \frac{\left((x_r + 13.5)x_r + 40\right)x_r + 16.67}{(3x_r + 27)x_r + 40}$$

we have the following table of results. (Note that the recurrence relation has been written in a form suitable for nested multiplication.)

r	$x_r\ (x_0 = -0.5)$	$x_r\ (x_0 = -3.5)$	$x_r\ (x_0 = -9.5)$
0	−0.5	−3.5	−9.5
1	−0.497 064 22	−3.546 760 6	−9.457 050 7
2	−0.497 068 01	−3.546 402 3	−9.456 529 8
3	−0.497 068 01	−3.546 402 3	−9.456 529 7

As you can see, after only 3 iterations the results agree to 7 or 8 significant figures. Hence, to 5 significant figures, the roots are −0.497 07, −3.5464 and −9.4565.

2. (i) The equation

$$xe^{-x} - 0.1 = 0$$

can be multiplied by $10e^x$, without introducing any new roots, to give

$$10x - e^x = 0.$$

Thus the original equation has roots whenever $10x = e^x$. Hence if we sketch the graphs of $y = 10x$ and $y = e^x$ the intersections of the two graphs will give the roots (see Figure 1).

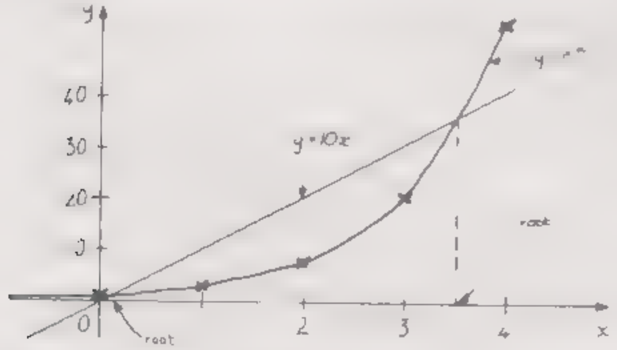


Figure 1. Sketch of $y = 10x$ and $y = e^x$.

There are two roots at approximately 0.2 and 3.5.

(ii) The Newton-Raphson method for this problem uses the recurrence relation

$$x_{r+1} = x_r - \frac{x_r \exp(-x_r) - 0.1}{(\exp(-x_r))(1 - x_r)},$$

i.e.

$$x_{r+1} = x_r - \frac{x_r - 0.1\exp(x_r)}{1 - x_r}.$$

r	$x_r(x_0 = 0.2)$	$x_r(x_0 = 3.5)$
0	0.2	3.5
1	0.102 675 34	3.575 381 9
2	0.111 744 32	3.577 151 1
3	0.111 832 55	3.577 152 1
4	0.111 832 56	3.577 152 1

To 6 decimal places the roots are 0.111 833 and 3.577 152.

3. The recurrence relation for the Taylor series method of order 1 is given by

$$Y_{r+1} = Y_r + hY_r'.$$

This is Euler's method (*Unit* 2, Subsection 2.2).

4. The recurrence relation for the Taylor series method of order 2 is given by

$$Y_{r+1} = Y_r + hY_r' + \tfrac{1}{2}h^2 Y_r''.$$

Now the differential equation is

$$y' = 3y + \sin x.$$

Differentiating gives

$$y'' = 3y' + \cos x.$$

Thus we have

$$Y_r' = 3Y_r + \sin x_r,$$
$$Y_r'' = 3Y_r' + \cos x_r.$$

r	x_r	Y_r	$Y_r' = 3Y_r + \sin x_r$	$Y_r'' = 3Y_r' + \cos x_r$	$Y_{r+1} = Y_r + hY_r' + \frac{1}{2}h^2 Y_r''$
0	0	0	[illegible]	1	0.02
1	0.2	0.02	0.258 669 33	1.756 074 6	0.106 855 36
2	0.4	0.106 855 36			

The approximate solutions at $x_1 = 0.2$ and $x_2 = 0.4$ are 0.02 and 0.106 855 36 respectively.

5. The recurrence relation for the Taylor series method of order 3 is given by

$$Y_{r+1} = Y_r + hY_r' + \tfrac{1}{2}h^2 Y_r'' + \tfrac{1}{6}h^3 Y_r'''.$$

The differential equation is

$$y' = \sin y.$$

Differentiating twice gives

$$y'' = y' \cos y,$$
$$y''' = y'' \cos y - y'^2 \sin y.$$

Hence the required equations are

$$Y_{r+1} = Y_r + hY_r' + \tfrac{1}{2}h^2 Y_r'' + \tfrac{1}{6}h^3 Y_r''', \text{ where}$$
$$Y_0 = 1$$
$$Y_r' = \sin Y_r$$
$$Y_r'' = Y_r' \cos Y_r$$
$$Y_r''' = Y_r'' \cos Y_r - Y_r'^2 \sin Y_r.$$

For $x_0 = 0$, $Y_0 = 1$ and $h = 0.1$ we have

$$Y_0' = \sin Y_0 = 0.841\,470\,99$$
$$Y_0'' = Y_0' \cos Y_0 = 0.841\,470\,99 \times \cos 1 = 0.454\,648\,72$$
$$Y_0''' = Y_0'' \cos Y_0 - Y_0'^2 \sin Y_0$$
$$= (0.454\,648\,72 \times \cos 1) - (0.841\,470\,99^2 \times \sin 1)$$
$$= -0.350\,175\,49$$

Thus

$$Y_1 = 1 + (0.1 \times 0.841\,470\,99) + (0.005 \times 0.454\,648\,72) - (\tfrac{1}{6} \times 0.001 \times 0.350\,175\,49)$$
$$= 1.086\,362.$$

Solutions to the exercises in Section 3

1. (i) $\cosh x = \dfrac{e^x + e^{-x}}{2}$. Thus,

$$\cosh(-x) = \frac{e^{-x}+e^{x}}{2} = \cosh x$$

$\sinh x = \dfrac{e^{x}-e^{-x}}{2}$. Thus,

$$\sinh(-x) = \frac{e^{-x}-e^{x}}{2} = -\sinh x.$$

The corresponding results for sines and cosines are identical, i.e.

$$\cos(-x) = \cos x \qquad \text{and} \qquad \sin(-x) = -\sin x.$$

(ii) $$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^{x}+e^{-x}}{2}\right) = \frac{e^{x}-e^{-x}}{2} = \sinh x,$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right) = \frac{e^{x}+e^{-x}}{2} = \cosh x.$$

The corresponding results for sines and cosines are almost identical, i.e.

$$\frac{d}{dx}(\sin x) = \cos x \qquad \text{and} \qquad \frac{d}{dx}(\cos x) = -\sin x.$$

2. Taylor's Theorem for $f(x) = \sinh x$ gives the $2n$th Taylor expansion about 0, with remainder, as

$$\sinh x = \sinh 0 + x\cosh 0 + \frac{x^2}{2!}\sinh 0 + \frac{x^3}{3!}\cosh 0 + \cdots + \frac{x^{2n}}{(2n)!}\sinh 0 + \frac{x^{2n+1}}{(2n+1)!}\cosh c_x$$

where c_x lies between 0 and x. (Use the result of Exercise 1 to do the differentiation.) With $\sinh 0 = 0$ and $\cosh 0 = 1$ we have

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n+1}}{(2n+1)!}\cosh c_x.$$

As n becomes large we obtain the required expansion for $\sinh x$, assuming (as is in fact true) that for fixed x the remainder term approaches zero for large n.

3. $y = \dfrac{1}{a}\cosh ax$. Thus,

$$\frac{dy}{dx} = \sinh ax.$$

Using the formula for the length of a curve, given in the box on p. 31:

$$\begin{aligned}
L &= \int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_{-1}^{1}\sqrt{1+\sinh^2 ax}\,dx \\
&= \int_{-1}^{1}\cosh ax\,dx \qquad (\text{since } \cosh^2 ax - \sinh^2 ax = 1) \\
&= \left[\frac{1}{a}\sinh ax\right]_{-1}^{1} \\
&= \frac{1}{a}(\sinh a - \sinh(-a)) \\
&= \frac{2}{a}\sinh a \qquad (\text{using the result of Exercise 1}).
\end{aligned}$$

4. Substituting $L = 2.08$ into Equation (7) gives

$$2.08 = \frac{2}{a}\sinh a,$$

which can be rearranged as

$$\sinh a - 1.04a = 0.$$

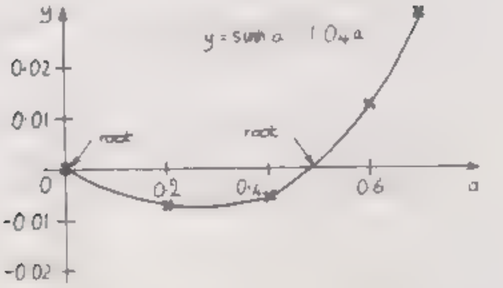


Figure 2. Sketch of $y = \sinh a - 1.04a$.

From the sketch graph of $y = \sinh a - 1.04a$ in Figure 2, we deduce that there is a root at approximately 0.49. The Newton-Raphson method gives

$$a_{r+1} = a_r - \left(\frac{\sinh a_r - 1.04a_r}{\cosh a_r - 1.04}\right).$$

$$\begin{aligned}
a_0 &= 0.49 \\
a_1 &= 0.487\,030\,33 \\
a_2 &= 0.487\,002\,63 = a_3.
\end{aligned}$$

Hence we have $a = 0.487\,002\,63$.

From Equation (6) we have

$$\begin{aligned}
d &= \frac{1}{a}(\cosh a - 1) \\
&= 0.248\,352\,16
\end{aligned}$$

Thus the displacement for $L = 2.08$ m is, to 6 significant figures:

$$d \simeq 0.248\,352\,\text{m}.$$

5. The 5th Taylor polynomial for $\sinh a$ about 0 is

$$p(x) = a + \frac{a^3}{6} + \frac{a^5}{120}.$$

Thus

$$\begin{aligned}
L = \frac{2}{a}\sinh a &\simeq \frac{2}{a}\left(a + \frac{a^3}{6} + \frac{a^5}{120}\right) \\
&= 2 + \frac{a^2}{3} + \frac{a^4}{60}.
\end{aligned}$$

Now to obtain a in terms of L in this approximation, we note that

$$L = 2 + \frac{a^2}{3} + \frac{a^4}{60}$$

is a quadratic in a^2, i.e.

$$\frac{a^4}{60} + \frac{a^2}{3} + (2 - L) = 0.$$

The formula method gives

$$\begin{aligned}
a^2 &= \frac{-\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{4}{60}(L-2)}}{1/30} \\
&= -10 \pm \sqrt{100 + 60L - 120} \\
&= -10 \pm \sqrt{60L - 20}.
\end{aligned}$$

As a^2 must be positive there is only one possibility, i.e.

$$a^2 = -10 + 2\sqrt{15L - 5}$$

giving

$$a = \sqrt{-10 + 2\sqrt{15L - 5}}.$$

6. If $L = 2.08$ m we have from Exercise 5, remembering that it is an approximation:

$$a \simeq 0.487\,018\,59.$$

Equation (6) gives

$$d = \frac{1}{a}(\cosh a - 1)$$
$$\simeq 0.248\,360\,61.$$

Comparing this with the accurate result in Exercise 4, we see that the approximation is correct to 4 significant figures, and even the fifth significant figure is nearly correct.

Solutions to the exercises in Section 4

1. Let the equation of the quadratic be

$$p(x) = a_0 + a_1 x + a_2 x^2.$$

Since $p(x)$ must pass through the three data points, we must have

$$\begin{aligned} p(-1) &= a_0 - a_1 + a_2 = 2 \\ p(0) &= a_0 = 3 \\ p(1) &= a_0 + a_1 + a_2 = 0. \end{aligned}$$

We can see that the solution is $a_0 = 3$, $a_1 = -1$ and $a_2 = -2$, so that

$$p(x) = 3 - x - 2x^2.$$

2. The interpolating polynomial is of the form

$$p(x) = a_0 + a_1 x.$$

The closest data points to $x = 1.134$ are $x = 1.13$ and $x = 1.14$. Thus we require $p(x)$ to satisfy the equations

$$\begin{aligned} p(1.13) &= a_0 + 1.13a_1 = 0.1222 \\ p(1.14) &= a_0 + 1.14a_1 = 0.1310. \end{aligned}$$

This gives $a_1 = 0.88$ and $a_0 = -0.8722$. Hence

$$p(x) = -0.8722 + 0.88x$$

and

$$p(1.134) = 0.125\,72.$$

3. (i) The maximum value of $|e^c|$ in the interval $-1 \leqslant c \leqslant 1$ occurs at $c = 1$ since e^x is a positive, increasing function in this interval. Hence in this interval,

$$0 \leqslant |e^c| \leqslant e = 2.7183 \qquad \text{to 4 decimal places.}$$

(ii) The equation for the error is given by

$$\varepsilon(x) = -(x - x_0)(x - x_1)(x - x_2)\frac{f'''(c_x)}{6}$$

where c_x lies between x_0 and x_2.

Now $f(x) = e^x$. Therefore $f'''(x) = e^x$. Hence,

$$\varepsilon(x) = -(x + 1)x(x - 1)\frac{\exp(c_x)}{6}.$$

Now for $-1 \leqslant c_x \leqslant 1$ we have $|\exp(c_x)| \leqslant 2.7183$, and so

$$|\varepsilon(x)| \leqslant |(x + 1)x(x - 1)|\frac{2.7183}{6}.$$

For $x = 0.5$ this gives

$$|\varepsilon(0.5)| \leqslant |1.5 \times 0.5 \times (-0.5)|\frac{2.7183}{6}$$
$$= 0.170 \qquad \text{to 3 decimal places.}$$

Thus an error bound for the interpolating polynomial approximation at $x = 0.5$ is 0.170.

4. Euler's method in composite form for the integral of e^x between $x = 0$ and $x = 1$ with $h = 0.2$ gives

$$\int_0^1 e^x dx \simeq 0.2(e^0 + e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8})$$
$$= 1.5522 \qquad \text{to 4 decimal places.}$$

(The true solution is $e - 1 = 1.7183$ to 4 decimal places.)

5. The composite trapezoidal method for the integral of e^x between $x = 0$ and $x = 1$ with $h = 0.2$ gives

$$\int_0^1 e^x dx \simeq 0.2(\tfrac{1}{2}e^0 + e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8} + \tfrac{1}{2}e^1)$$
$$= 1.7240 \qquad \text{to 4 decimal places.}$$

The true solution is $e - 1 = 1.7183$ to 4 decimal places. The result is much more accurate than the result obtained using the composite Euler's method.

6. (i) To use Simpson's method to approximate the integral of e^x between $x = 0$ and $x = 1$ we have $h = 0.5$, giving

$$\int_0^1 e^x\,dx \simeq \frac{0.5}{3}(e^0 + 4e^{0.5} + e^1)$$
$$= 1.7189 \qquad \text{to 4 decimal places.}$$

This is much more accurate, for much less work, than the approximations in Exercises 4 and 5.

(ii) With $h = \frac{1}{4}$ we have

$$\int_0^1 e^x dx \simeq \frac{0.25}{3}(e^0 + 4e^{0.25} + 2e^{0.5} + 4e^{0.75} + e^1)$$
$$= 1.718\,318\,8.$$

(The true solution is 1.718 281 8.)

With $h = \frac{1}{8}$ we have

$$\int_0^1 e^x \simeq \frac{0.125}{3}(e^0 + 4e^{0.125} + 2e^{0.25} + 4e^{0.375} + 2e^{0.5} + 4e^{0.625} + 2e^{0.75} + 4e^{0.875} + e^1)$$
$$= 1.718\,284\,2 \qquad \text{to 7 decimal places.}$$

This is an extremely accurate result.

7. The approximate value for

$$L = \int_0^{\pi/2} (1 + \cos^2 x)^{1/2}\,dx$$

using the composite Simpson's method with $h = \dfrac{\pi}{8}$ is

$$L \simeq \frac{\pi}{24}\left\{(1 + \cos^2 0)^{1/2} + 4\left(1 + \cos^2\frac{\pi}{8}\right)^{1/2} + 2\left(1 + \cos^2\frac{\pi}{4}\right)^{1/2} + 4\left(1 + \cos^2\frac{3\pi}{8}\right)^{1/2} + \left(1 + \cos^2\frac{\pi}{2}\right)^{1/2}\right\}$$
$$= 1.910\,141 \qquad \text{to 6 decimal places.}$$

(A more accurate answer, obtained by computer, using Simpson's method with $h = \pi/200$, is 1.910 099 to 6 decimal places. The solution obtained is thus correct to 4 decimal places.)

Solutions to the exercises in Section 5

1. (i) From the hint we know that

$$\begin{aligned} 2\tan x \sec^2 x &= 2\tan x\,(1 + \tan^2 x) \\ &= 2\tan x + 2\tan^3 x. \end{aligned}$$

Since $\tan x$ is positive and increasing in the interval $0 \leqslant x \leqslant 0.1$, the maximum value of the above function occurs at $x = 0.1$. Hence,

$$2 \tan c \sec^2 c \leqslant 2 \tan 0.1 \sec^2 0.1$$
$$= 0.20 \quad \text{to 2 decimal places.}$$

(ii) The linear Taylor polynomial about 0 is given by

$$p(x) = f(0) + xf'(0).$$

For $f(x) = \tan x$ we have

$$f'(x) = \sec^2 x = \frac{1}{\cos^2 x},$$

so that $f(0) = 0$ and $f'(0) = 1$.

Hence the linear Taylor polynomial about 0 approximating $\tan x$ is given by

$$p(x) = x.$$

The error function is given by

$$\varepsilon(x) = -\tfrac{1}{2}x^2 f''(c_x)$$

where c_x lies between 0 and x.

Now $f''(x) = \dfrac{d}{dx}\sec^2 x = 2 \tan x \sec^2 x$.

Hence $0 \leqslant f''(c_x) = 2 \tan c_x \sec^2 c_x$
$\leqslant 0.20$ by part (i), since $0 \leqslant c_x \leqslant 0.1$.

A bound for $\varepsilon(x)$ is therefore

$$|\varepsilon(x)| \leqslant \tfrac{1}{2}x^2 \times 0.20 = 0.1x^2.$$

2. The Newton-Raphson method uses the recurrence relation

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}.$$

We have

$$f(x) = e^x - 3x, \quad f'(x) = e^x - 3.$$

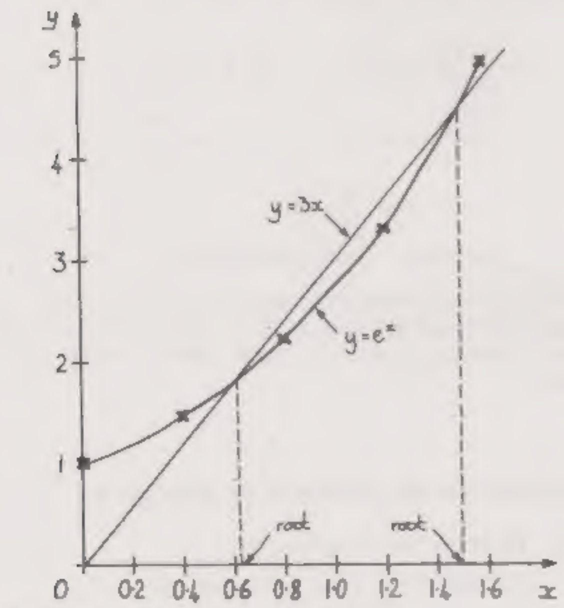


Figure 3. Graphs of $y = e^x$ and $y = 3x$.

Hence for this problem the recurrence relation is

$$x_{r+1} = x_r - \frac{\exp(x_r) - 3x_r}{\exp(x_r) - 3}. \tag{1}$$

To obtain an initial approximation we sketch the graphs of $y = e^x$ and $y = 3x$ as in Figure 3.

From the graph we can see that two curves intersect at approximately $x = 0.6$ and $x = 1.5$. With $x_0 = 0.6$, Equation (1) above gives

$$x_1 = 0.618\,778\,47$$
$$x_2 = 0.619\,061\,22$$
$$x_3 = 0.619\,061\,29$$
$$= x_4.$$

With $x_0 = 1.5$, Equation (1) gives

$$x_1 = 1.512\,358\,1$$
$$x_2 = 1.512\,134\,6$$
$$= x_3.$$

Hence to 5 decimal places the two roots are 0.619 06 and 1.512 13.

3. The Taylor series method of order 2 uses the recurrence relation

$$Y_{r+1} = Y_r + hY_r' + \tfrac{1}{2}h^2 Y_r'', \text{ where}$$
$$Y_r' = x_r Y_r + 1 \quad \text{and}$$
$$Y_r'' = Y_r + x_r Y_r'.$$

With $h = 0.1$ we obtain the following table of results.

r	x_r	Y_r	$Y_r' = x_rY_r + 1$	$Y_r'' = Y_r + x_rY_r'$	$Y_{r+1} = Y_r + 0.1Y_r' + 0.005Y_r''$
0	0	1	1	1	1.105
1	0.1	1.105	1.1105	1.21605	1.222 130 3
2	0.2	1.222 130 3			

Thus: when $x = 0.1$, $y \simeq 1.105$,
when $x = 0.2$, $y \simeq 1.222$.

4. (i) Let

$$p(x) = a + bx$$

be the interpolating polynomial.

Since the graph of $y = p(x)$ passes through the points $(1, 1)$ and $(2\frac{1}{4}, 1\frac{1}{2})$, we have

$$p(1) = a + b = 1$$
$$p(2\tfrac{1}{4}) = a + 2\tfrac{1}{4}b = 1\tfrac{1}{2}.$$

Solving for a and b gives $a = \frac{3}{5}$ and $b = \frac{2}{5}$. Hence

$$p(x) = 0.6 + 0.4x.$$

At $x = 1.44$ we have

$$p(1.44) = 1.176.$$

(ii) Since $\frac{1}{4}x^{-3/2}$ is a positive decreasing function on the interval $1 \leqslant x \leqslant 2\frac{1}{4}$, its maximum value is at $x = 1$, and is equal to $\frac{1}{4}$. Hence

$$|\tfrac{1}{4}c^{-3/2}| \leqslant \tfrac{1}{4} \qquad \text{for } 1 \leqslant c \leqslant 2\tfrac{1}{4}.$$

(iii) From Theorem 4.1 on page 39, the error function is given by

$$\varepsilon(x) = -(x-1)(x-2\tfrac{1}{4})\frac{f''(c_x)}{2}.$$

Now $f(x) = \sqrt{x}$, and so

$$f'(x) = \tfrac{1}{2}x^{-1/2} \quad \text{and} \quad f''(x) = -\tfrac{1}{4}x^{-3/2}.$$

From part (ii) we know that an upper bound for $|f''(c_x)|$ where $1 \leqslant c_x \leqslant 2\frac{1}{4}$ is given by $M = \frac{1}{4}$. Hence at $x = 1.44$,

$$|\varepsilon(1.44)| \leqslant \frac{|(1.44-1)(1.44-2.25)|}{8} = 0.044\,55.$$

The actual error at $x = 1.44$ is

$$\begin{aligned}\varepsilon(1.44) &= p(1.44) - \sqrt{1.44} = 1.176 - 1.2\\ &= -0.024\end{aligned}$$

and its magnitude is less than the error bound for the approximation.

5. Simpson's method in composite form with $h = 0.25$, approximating the integral of $\tan x$ between 0 and 1, gives

$$\begin{aligned}\int_0^1 \tan x\,dx &\simeq \frac{0.25}{3}(\tan 0 + 4\tan 0.25 + 2\tan 0.5\\ &\qquad + 4\tan 0.75 + \tan 1)\\ &= 0.616\,480\,52.\end{aligned}$$

(The true solution is

$$\int_0^1 \tan x\,dx = [\log_e(\sec x)]_0^1 = 0.615\,626\,47.)$$

6. From Section 3 the length L of a curve y is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

For $y = e^x$ we have $\dfrac{dy}{dx} = e^x$ and

$$L = \int_0^1 \sqrt{1 + e^{2x}}\,dx.$$

Simpson's method for this integral, with $h = 0.25$, gives

$$\begin{aligned}L &\simeq \frac{0.25}{3}(\sqrt{2} + 4\sqrt{1+e^{1/2}} + 2\sqrt{1+e} + 4\sqrt{1+e^{3/2}}\\ &\qquad + \sqrt{1+e^2}) = 2.003\,527\,5.\end{aligned}$$

The Open University
0 335 14047 5